# *Dynamics in chaotic zones (APMs): inner/outer splittings of separatrices*

*Workshop on instabilities in Hamiltonian systems*

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# Introduction: general framework

Let $F_\nu$ be a one-parameter family of APMs

$F_\nu(E_0) = E_0$ elliptic fixed point,

Spec$(DF_\nu)(E_0) = \{\lambda, \lambda^{-1}\}$, $\lambda = \exp(2\pi i \alpha)$.

$\rightarrow$ Assume that we are interested in the dynamics close to the $(q\!:\!m)$-resonance for $q, m \in \mathbb{N}$, with $1 \leq q < m$, gcd$(q,m) = 1$. Then, one can write $\alpha = q/m + \delta$ with $\delta \in \mathbb{R}$ (generically $\alpha'(\nu) \neq 0$) and we denote the family as $F_\delta$ (for arbitrary $q$ and $m$).

$\rightarrow F_\delta : \mathcal{U} \rightarrow \mathbb{R}^2, \mathcal{U} \subset \mathbb{R}^2$ domain, is such that

1. $F_\delta$ real analytic in the $(x,y)$-coordinates of $\mathcal{U}$,
2. $\det DF_\delta(x,y) = 1$, for all $(x,y) \in \mathbb{R}^2$ and for all $\delta \in \mathbb{R}$, (APMs)
3. $F_\delta$ has a fixed point $E_0$ that will be assumed to be at the origin $\forall \delta \in \mathbb{R}$,
4. spec $DF(E_0) = \{\mu, \bar{\mu}\}$, $\mu = \exp(2\pi i \alpha)$, $\alpha = q/m + \delta$, $q, m \in \mathbb{Z}$.

# Hénon map

As an example consider the Hénon map

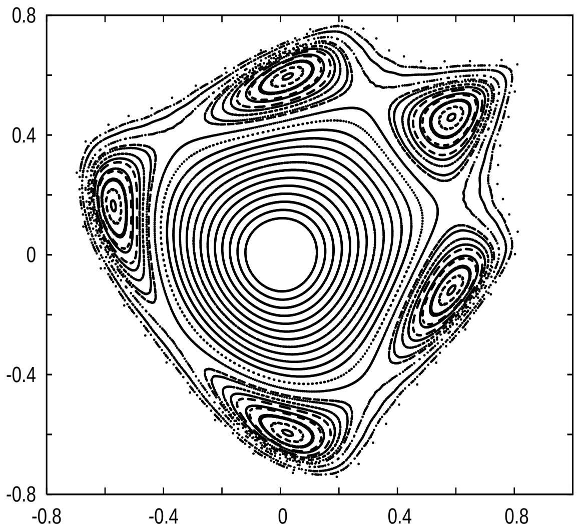
$$H_\alpha(x,y) = R_{2\pi\alpha}(x, y - x^2), \qquad \alpha \in (0, 1/2)$$

- It has two fixed points:

  the origin is an elliptic fixed point $E_0$,

  the point $P_h = (2\tan(\pi\alpha), 2\tan^2(\pi\alpha))$ is a hyperbolic fixed point.

- Reversible with respect to $y = x^2/2$ and $y = \tan(\pi\alpha)x$.



# Goal of our study of APMs

- We want to describe the dynamics in the **resonant chains** emanating from (but **relatively far** from) the elliptic fixed point $E_0$.
- Special interest in the **dynamics in chaotic zones**, either small regions around the separatrices of the resonant island (with size of the order of the splitting of the separatrices) or the larger regions due to interaction of different resonances (for example, Birkhoff zones of instability if the region is confined between rotational invariant curves).
- We are interested in a topological/qualitative description but **our goal is to obtain quantitative information** of the system: size of the chaotic zones, distance to invariant curves, measure of the stability regions within the chaotic zones, transport properties,...
- General strategy: BNF $\rightarrow$ Interp. Hamiltonian $\rightarrow$ Simplified Model

# BNF

$F_\delta$ one-parameter $\delta$-family of APMs with $F(E_0) = E_0$ elliptic fixed point.

Spec $DF(E_0) = \{\mu, \bar{\mu}\}$, $\mu = e^{2\pi i \alpha}$, $\alpha = q/m + \delta$, $\delta$ small enough.

$(x, y)$-cartesian coord., $(z, \bar{z})$-complex coord. ($z = x + iy$, $\bar{z} = x - iy$).

The Birkhoff NF to order $m$ around $E_0$ can be expressed as

$$\text{BNF}_m(F)(z) = R_{2\pi \frac{q}{m}} \left( \underbrace{e^{2\pi i \gamma(r)} z}_{\text{unavoidable res.}} + \underbrace{i \bar{z}^{m-1}}_{m\text{-order res.}} \right) + R_{m+1}(z, \bar{z}),$$

where

$$\gamma(r) = \delta + b_1 r^2 + b_2 r^4 + \ldots + b_s r^{2s}, \quad r = |z|,$$

being

$s = [(m-1)/2]$,

$b_i \in \mathbb{R}$ are the so-called Birkhoff coefficients,

$R_{m+1}(z, \bar{z})$ denotes the remainder which is of $\mathcal{O}(m+1)$.

# Interpolating flow of the BNF

$(I, \varphi)$-Poincaré variables ($z = \sqrt{2I}\exp(i\varphi)$).

$$\mathcal{H}_{nr}(I) = \pi \sum_{n=0}^{s} \frac{b_n}{n+1}(2I)^{n+1} \quad \text{and} \quad \mathcal{H}_r(I,\varphi) = \frac{1}{m}(2I)^{\frac{m}{2}}\cos(m\varphi).$$

Let $r_*$ such that $\gamma(r_*) = 0$, that is $r_* \approx (-b_0/b_1)^{1/2}$, $b_0 = \delta$.

$\rightarrow$ The flow $\phi$ generated by the Hamiltonian

$$\mathcal{H}(I,\varphi) = \mathcal{H}_{nr}(I) + \mathcal{H}_r(I,\varphi)$$

interpolates $K$ with an error of order $m+1$ with respect to the $(z, \bar{z})$-coordinates, that is,

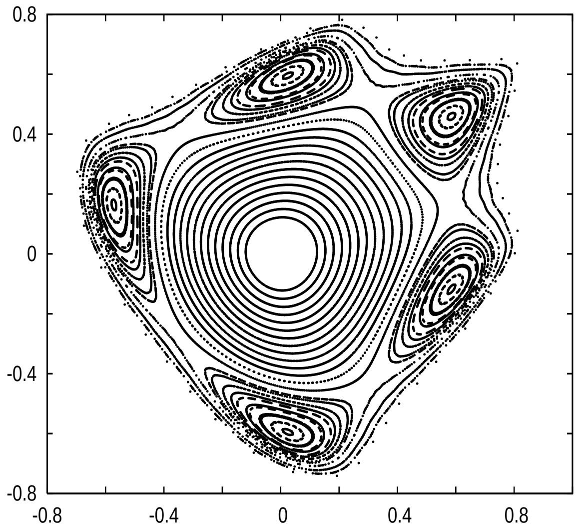
$$K(I,\varphi) = \phi_{t=1}(I,\varphi) + \mathcal{O}\left(I^{\frac{m+1}{2}}\right).$$

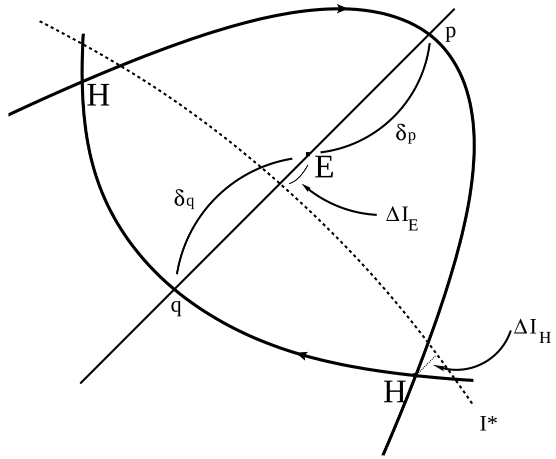
If we assume $b_1 \neq 0$ this approximation holds in an annulus centred in the resonance radius $r_*$ of width $r_*^{1+\nu}$, for $\nu > 0$.

# *Description of resonances*

Generic case: $\alpha = q/m + \delta$, $m > 5$, $\delta$ sufficiently small, $b_1 \neq 0$.

- If $b_1\delta < 0$ then $F$ **has** a resonant island of order $m$.
- The resonant zone is determined by **two periodic orbits** of period $m$ located near two concentric circumferences (in the BNF variables). The closest orbit to the external circumference is elliptic while the one located close to the inner circumference is hyperbolic.
- The **width** of the resonant island is $\mathcal{O}(I_*^{m/4})$, $I_* = -\delta/2b_1$.




"Outer splitting $\leftrightarrow$ p"

"Inner splitting $\leftrightarrow$ q"

# A model around a generic resonance

For a generic APM such that $\delta < 0,\ b_1 > 0,\ b_2 \neq 0$, the dynamics around an island of the $m$-resonance strip $(m \geq 5)$ can be modeled, after suitable scaling ($J \sim \delta^{-m/4}(I - I_*)$), by the time-$\log(\lambda)$ map of the flow generated by

$$\mathcal{H}(J,\psi) = \frac{1}{2}J^2 + \frac{c}{3}J^3 - (1 + dJ)\cos(\psi),$$

where $c = \mathcal{O}(\delta^{\frac{m}{4}}),\ d = \mathcal{O}(\delta^{\frac{m}{4}-1})$. Bounding the errors, it is shown that it gives a “good” enough approximation of the dynamics in an annulus containing the $m$-islands.

$\longrightarrow$ Then, we have the following... [a]

[a] The details of the proof (singularities, suitable Hamiltonian,...) can be found in:
*Resonant zones, inner and outer splittings in generic and low order resonances of area preserving maps.* Nonlinearity 22, 5:1191–1245, 2009.

# Main result: comments on the hypothesis

- **A1.** $b_1(\delta)$ is non-zero for $\delta = 0$.
- **A2.** $P_h^r$ – hyperbolic $m$-periodic point on a resonant zone close to $E_0$,
  $\gamma(t)$ – separatrix of the interp. Hamiltonian flow $\varphi_t$,
  Assume that the closest singularities of $\gamma(t)$ to $\mathbb{R}$ have $|\text{Im}(t)| = \tau$.
  Represent $W^u_{P_h^r}$ and $W^s_{P_h^r}$ as functions of $t$, close to $\gamma(t)$.
  $\mathcal{E}(t)$ – distance $W^u_{P_h^r}(t) - W^s_{P_h^r}(t)$ (periodic in $t$).
  $G(t)$ – restriction of $\mathcal{E}(t)$ to $t + i(\tau - \mathcal{O}(\delta^q))$, $t \in \mathbb{R}$, $q > 0$.
  We require that there exist constants $k_1, k_2 > 0$ and $j_2 \leq j_1$ such that for all $\delta$, $0 < \delta < \delta_0$, one has $k_1\delta^{j_1} < |G| < k_2\delta^{j_2}$ and that the first harmonic $c_1$ of the Fourier expansion of $G(t)$ verifies $|c_1| > \alpha|G|$, with $\alpha > 0$ a constant independent of $\delta$.
- **A3.** $F$ maybe meromorphic but the possible singularities remain at a finite distance as $|\delta| \searrow 0$.

# *Main result*

**Theorem.** Let $F$ be an APM. Assume that it has an $m$-order resonance strip, $m > 4$, located at an average distance $I = I_* = \mathcal{O}(\delta)$ from the elliptic fixed point and $\delta$ is sufficiently small. Under the assumptions **A1**, **A2** and **A3**, the following conclusions hold.

a) The outer splitting is larger than the inner one being the difference between the position of the corresponding nearest singularities $\mathcal{O}(\delta^{m/4-1})$.

b) Neither the inner nor the outer splittings oscillate.

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$\longrightarrow$ **Comment:** It does not apply to strong resonances (e.g. the 1:4 resonance of the Hénon map), and it does not apply if too far from the origin (e.g. the 2:11 resonance of the Hénon map).

$\longrightarrow$ **Question:** Consequences on the width of the chaotic zones of this fact?

General strategy: Glue different simple universal return models.

# *Return models & chaotic regions considered*

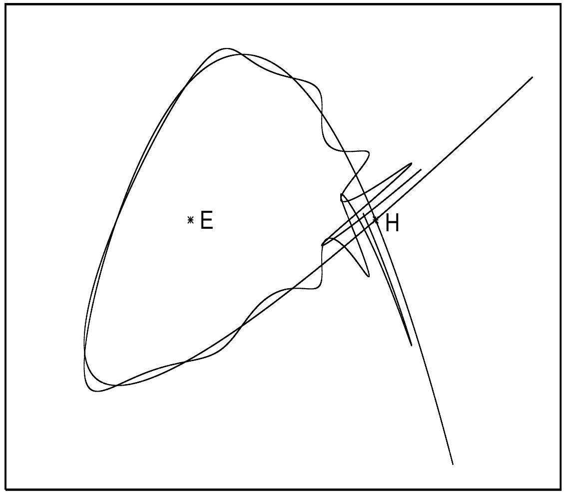
For each of the following cases we use a concrete return map model to study the dynamics.

- Open case (fish like) — Separatrix map
- Figure eight case (pendulum like) — Double separatrix map
- Large regions of instability (e.g. Birkhoff z.i.) — Biseparatrix map

$\longrightarrow$ We look for **quantitative** information on the dynamics within the chaotic zones. However, the biseparatrix model only gives us a topological description of the dynamical behaviour. [a]

[a] The following results can be found in:
*Dynamics in chaotic zones of area preserving maps: close to separatrix and global instability zones.*
Physica D, 240(8), 2011.

# Open case




$$SM : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + a + b\log|y'| \\ y + \sin(2\pi x) \end{pmatrix}$$

where $b = 1/\log(\lambda)$, $\lambda$ the dominant eigenvalue of $DF(h)$ and $a$ is a “shift”.

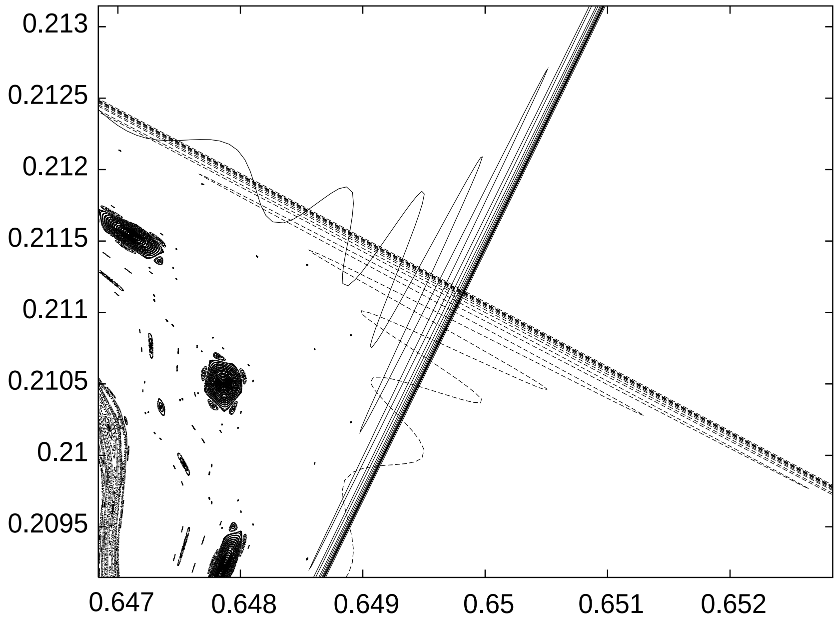
The $y$-vble. is scaled by the amplitude of the splitting.

We deal with an **a priori stable** case: $\log(\lambda) = \mathcal{O}(\epsilon)$ and $a = \mathcal{O}(1/\epsilon) \Rightarrow A = \mathcal{O}(\exp(-ctant/\epsilon^r)$. Here $\epsilon$ is a “distance-to-integrable” parameter.

# *Open case: results*

- Distance to invariant curves from the separatrix: $d_c \sim |b|/k^*$ (SM is approximated by STM, $k^* \approx 0.97/(2\pi)$ Greene value).
  - ▶ When coming back to the original variables: $D_c \sim \sigma\ell/(2\pi k^* \log(\lambda))$,
  - ▶ If measured from the hyperbolic point, assuming the map close to the time-$\epsilon$ flow of $H(x,y) = y^2/2 - \alpha x^3 - \beta x^2$, one has: $D_c^h \approx (3LD_c/2)^{1/2}$, where $L$ is the distance between the hyperbolic and the elliptic point inside the "fish". This result can be improved using higher order interpolating Hamiltonians.
- Distance to islands from the separatrix: $d_i \sim |b|/\tilde{k}$, $\tilde{k} = 2/\pi$.
- Expected number of "central" islands before the r.i.c. $\#\{islands\} \approx 1.415 \times b$.

# *Hénon map* $H_\alpha(x,y) = R_{2\pi\alpha}(x, y - x^2)$



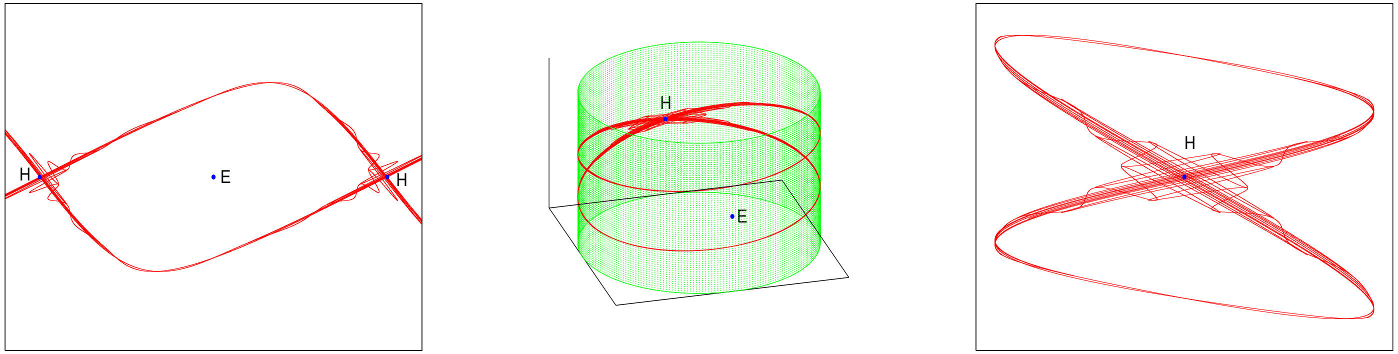

$\alpha = 0.1$

Experimental values: $(D_c^H)_e \approx 2.94 \times 10^{-3}$, $(D_i^H)_e \approx 2.08 \times 10^{-3}$

“Fish” interpolating Hamiltonian: $D_c^H \approx 2.47 \times 10^{-3}$, $D_i^H \approx 1.85 \times 10^{-3}$

5-order interp. Hamiltonian: $D_c^H \approx 2.731 \times 10^{-3}$, $D_i^H \approx 2.050 \times 10^{-3}$

# *Figure eight case*




$$DSM : \begin{pmatrix} x \\ y \\ s \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} x + a_{\bar{s}} + b \log |\bar{y}| \pmod 1 \\ y + \nu_{\bar{s}} \sin 2\pi x \\ \text{sign}(y)\, s \end{pmatrix},$$

where $\nu$ is such that $\nu_1 = 1$ and $\nu_{-1} = A_{-1}/A_1$, being $A_1$ and $A_{-1}$ the amplitudes of the outer and inner splittings, respectively, of the resonant island.

**Comments:**

- It is defined on a domain $\mathcal{W} = \mathcal{U} \cup \mathcal{D}$ (upper and lower domains around the outer and inner separatrices of the resonance).
- $y > 0$ means we are outside the stable manifold (either in $\mathcal{U}$ or $\mathcal{D}$).

# Figure eight: results

**Generic resonances close to the origin.** Assume $b_1\delta < 0$ and that the hypothesis of the theorem concerning the difference of the inner and outer splittings hold. Then,

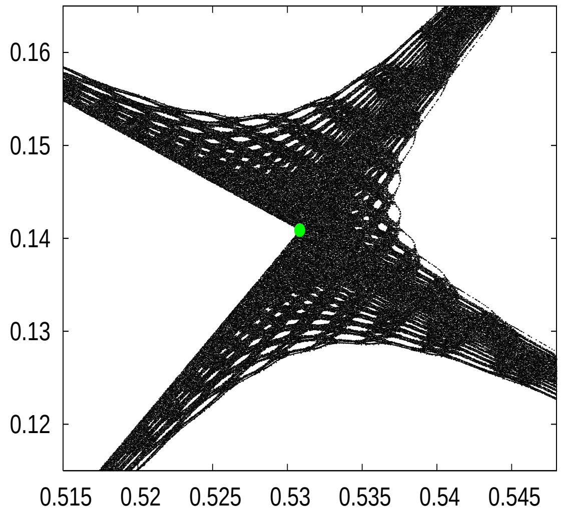
- The width of the outer chaotic zone is larger than the width of the inner chaotic one if, and only if, sign $b_1 \cdot$ sign $b_2 < 0$.
- Both amplitudes of the stochastic layer are of the order of magnitude of the outer splitting (the largest one).

# Pendulum-like islands: comments

The idea is to construct an interpolating Hamiltonian of the map (in a domain containing the resonance) and to use preservation of energy to see how the distance to the rotational invariant curves changes when measuring from the upper $\mathcal{U}$ and the lower $\mathcal{D}$ domains. This can be done computing the ratio

$$f = \nabla\mathcal{H}(J_M)/\nabla\mathcal{H}(J_m)$$

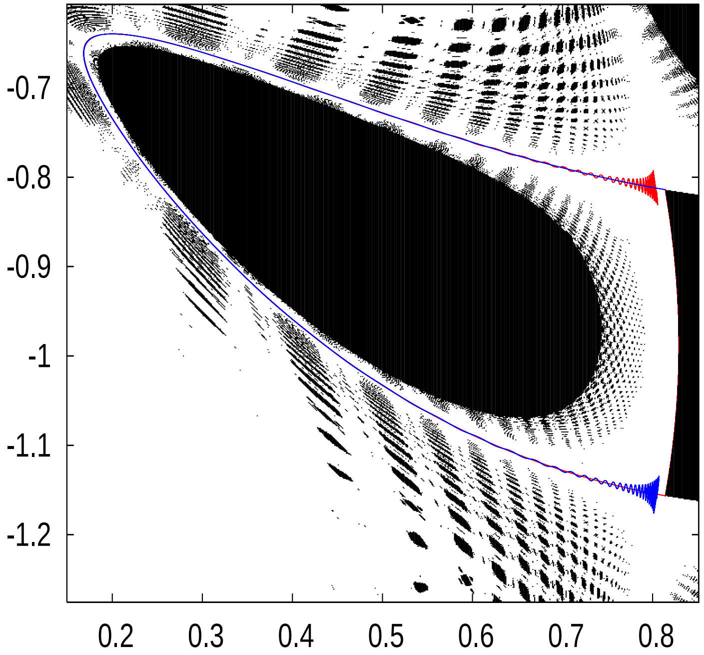
where $J_M$ and $J_m$ are the maximum (minimum) of the outer (inner) separatrix of the Hamiltonian. For close to the origin resonances $f = 1 + \mathcal{O}(\delta^{m/4})$.




$$\alpha = 0.21,$$
$$\sigma_- = \mathcal{O}(10^{-12}),$$
$$\sigma_+ = \mathcal{O}(10^{-3}).$$

# *Pendulum-like islands: comments II*

The same idea applies to resonances far from the origin as well as for strong resonances but, for each case, a suitable interpolating Hamiltonian must be considered. In these cases the chaotic zone width measured in both domains can be of different order of magnitude:




$c = 1.015$,

$\sigma_+ = \mathcal{O}(10^{-54})$, $\sigma_- = \mathcal{O}(10^{-1})$.

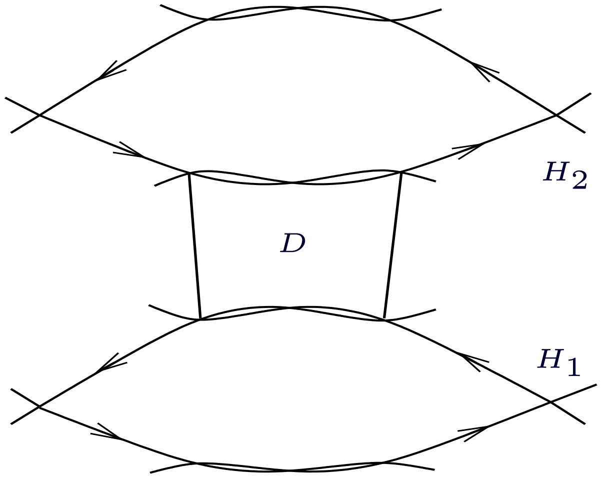
Experimentally, $f \approx -5$. Using interp. Ham. up to order $\delta \approx c - 1$ we obtain $f \approx -5.64$.

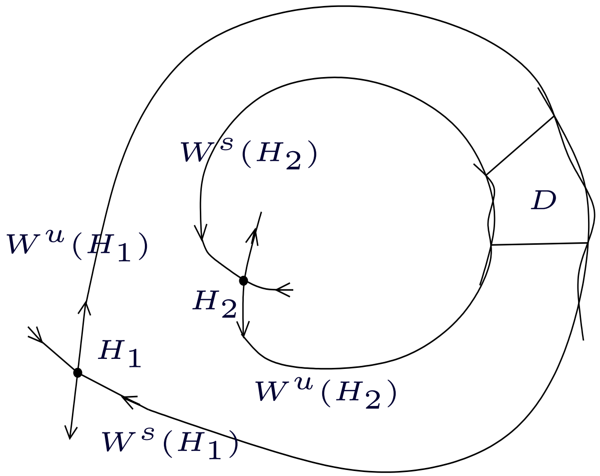
But $\delta = 0.015$ is too large. For $\delta$ small we obtain better results (even we can predict # tiny islands).

# *Large regions of stability*

Due to the interaction of resonances large chaotic zones of instability appear. These are regions without rotational invariant curves (e.g. Birkhoff zones of instability). We have considered the **biseparatrix model** and we have studied different situations (twist and non-twist case). On the other hand, it helps to study the phenomena taking place at the border of the stability domain.

Geometrical situation:



# The biseparatrix model

Between two concentric chains of islands, the simplest *qualitative* model on the domain $0 < v < d$ is given by

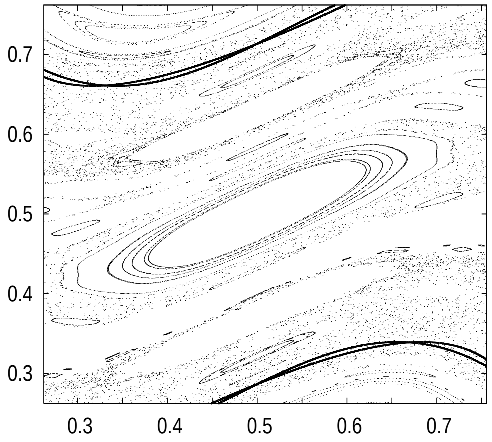
$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u + \alpha + \beta_1 \log(v') - \beta_2 \log(d - v') \\ v + \sin(2\pi u) \end{pmatrix}.$$

where $\beta_1 = 1/\log(\lambda)$, $\beta_2 = 1/\log(\mu)$, $\lambda$ being the eigenvalue of modulus greater than one of the hyperbolic point of the bottom separatrix and $\mu$ the corresponding one of the top separatrix.

- For this model it is theoretically expected to have rotational invariant curves provided $d > (\sqrt{b_1} + \sqrt{b_2})^2/k^*$ ($k^* =$ Greene's value).
- Changing - for + in the 1st. row it is a model for non-twist Birkhoff zones.
- It remains to generalise it to different order for the top/bottom resonances and to make it quantitative.

# *BSM figures*

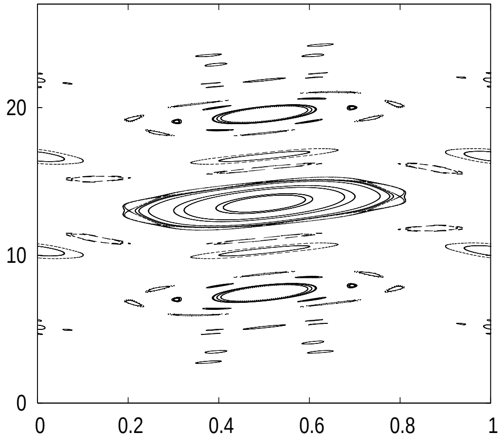
Chirikov standard map with $k = 0.16$.

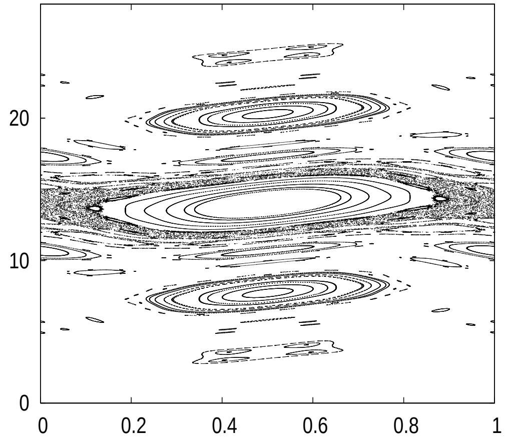



For the corresponding BSM model:

$\beta_1 = \beta_2 \approx 1.0365$

$(\lambda = \mu \approx 2.624248)$




$$\left.\begin{array}{r} \text{Amplitude inner/outer splitting: } A \approx 6 \times 10^{-3} \\ \text{Islands distance: } \hat{d} \approx 0.426987 \\ \text{Adjacent homoclinics distance: } a \approx 0.017796 \end{array}\right\} \Rightarrow d \approx 24\ (d = 27, 28)$$

# *Work in progress...*

We are working in 4D symplectic maps trying to generalise these studies (in collaboration with V. Gelfreich and E. Fontich).

The main goal is to get a qualitative/quantitative description of the dynamics in **double resonances**.

In the following we present some preliminary results in this direction.

# 4D symplectic maps

We consider a two parametric family of 4D symplectic maps $F_\delta$, wich depend on a (small) vector $\delta \in \mathbb{R}^2$. We study the dynamics of $F_\delta$ around a totally elliptic fixed point which is assumed to be at the origin. That is,

$$F_\delta(0) = 0,$$

$$DF_\delta(0) = \{\lambda_1, \bar{\lambda_1}, \lambda_2, \bar{\lambda_2}\}, \ \lambda_j = e^{2\pi i \alpha_j}, \ \alpha_j \in (0, 1/2), j = 1, 2.$$

The local dynamics can be described using **BNF**, which depends on arithmetic properties of $\alpha_j$. The subgroup $\Gamma \subset \mathbb{Z}^2$ will denote the set of resonances,

$$\Gamma = \{\, (k_1, k_2) : k_1\alpha_1 + k_2\alpha_2 = 0 \pmod 1 \,\}$$

Note that $(k_1, k_2) \in \Gamma \Leftrightarrow \lambda_1^{k_1}\lambda_2^{k_2} = 1$.
We say that $\mathbf{r} = (k_1, k_2) \in \Gamma$ is *a resonance of order* $|\mathbf{r}| = |k_1| + |k_2|$.
We assume $k_1 \geq 0$ to avoid trivial symmetries in resonances.

# Resonances

The fixed point can be:

1. **Non-resonant** ($\Gamma$ is a trivial group): $\{\, \alpha_1, \alpha_2, 1 \,\}$ are rationally independent.
2. **Simply resonant** ($\Gamma$ is a 1D lattice). Two possibilities:
   a) $\alpha_1 \in \mathbb{Q}$, $\alpha_2 \in \mathbb{R} \setminus \mathbb{Q}$.
   b) $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \mathbb{Q}$ but $\{1, \alpha_1, \alpha_2\}$ are rationally dependent.
3. **Doubly resonant** ($\Gamma$ is a 2D lattice): $\alpha_j = p_j/q_j$, $p_j, q_j \in \mathbb{N}$, $j = 1, 2$ (irred. fractions).

$\longrightarrow$ Interest in dynamics close to **double resonances**, hence we assume $\alpha_j = p_j/q_j + \delta_j$, $\delta_j$ small. Moreover, we consider $\alpha_1 \neq \alpha_2$ (simple eigenvalues) for all $\delta = (\delta_1, \delta_2)$.

# *Primary resonances*

Let $\mathbf{r}_0 \in \mathbb{Z}^2$ be the smallest non-trivial element of $\Gamma$ and $\mathbf{r}_1 \in \mathbb{Z}^2$ be the smallest element independent from $\mathbf{r}_0$. It can be proved that $\mathbf{r}_0$, $\mathbf{r}_1$ generate $\Gamma \subset \mathbb{Z}^2$ provided $\alpha_1 \neq \alpha_2$. We call $\mathbf{r}_0$, $\mathbf{r}_1$ the **primary resonances** (minimal generators of $\Gamma$).

We denote by $n_j$ the order of $\mathbf{r}_j$, $j = 1, 2$ (then, $n_0 \leq n_1$).

**Remark:** The **primary resonances** are unique *for most* of the frecuencies. However, there are two situations of non-uniqueness for specific resonances:

- At the leading order. Consider $\alpha_1 = 1/8$, $\alpha_2 = 3/8$. Then $n_0 = n_1 = 4$ and there are 3 resonances of order $4$: $(1, -3)$, $(3, -1)$, $(2, 2)$.
- At order $n_1$. Consider $\alpha_1 = 1/11$, $\alpha_2 = 4/11$. Then $n_0 = 4$ and the only resonance of order 4 is $(1, -3)$, and $n_1 = 5$ and there are 2 resonances: $(4, -1)$ and $(3, 2)$.

# Classification by primary resonances

If $n_0 \geq 5$ the fixed point is called *weakly resonant* and otherwise it is *strongly resonant*. There are different situations to study:

- $5 \leq n_0 < n_1$: Up to order $n_0 - 1$ the interp. Hamiltonian of BNF looks like the one of a non-resonant point. Moreover, up to order $n_1 - 1$ is also integrable.
- $5 \leq n_0 = n_1$: Up to order $n_0 - 1$ the interp. Hamiltonian of BNF looks like the one of a non-resonant point. Adding order $n_0$-terms becomes non-integrable.
- Strong resonances ($n_0 \leq 4 < n_1$) and doubly strong resonances ($n_0 = 3$, $n_1 = 4$, and $n_0 = n_1 \leq 4$).

# The simplest case

For frequencies $\alpha_1 = p_1/q_1$ and $\alpha_2 = p_2/q_2$ such that $5 \leq n_0 < n_1$ the following model (two coupled pendulums), obtained from interp. Hamiltonian of BNF, describes the dynamics in the corresponding double resonance:

$$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + \cos(\psi_1) + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} + \epsilon \cos(\psi_2),$$

where $\epsilon = \mathcal{O}(\delta^{(q_2 - q_1)/2})$, $a_2, a_3 \in \mathbb{R}$.

- For $\epsilon = 0$, $\{\psi_1 = 0, J_1 = 0\}$ NHIM $\Rightarrow$ a priori unstable system.
- It has 4 fixed points: $EE$, $EH$, $HE$ and $HH$ (for suitable $a2, a3, \epsilon$ parameters the $EE$ point becomes complex saddle).
- Slow-fast structure: $(\psi_1, J_1)$ "fast" pendulum, $(\psi, J_2)$ "slow" pendulum.
- Non-integrable: The splitting of the 2D invariant manifolds of the $H - H$ point (of the Hamiltonian system) is $\mathcal{O}(\sqrt{\epsilon})$.

**Thanks for your attention!!**