The transition to complex-saddle in a Froeschlé-type map

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A. Vieiro

Joint work with E. Fontich and C. Simó

University of Barcelona (UB)

Motivation: A paradigmatic Froeschlé-like map

Consider the map $T : (\psi_1, \psi_2, J_1, J_2) \mapsto (\bar{\psi}_1, \bar{\psi}_2, \bar{J}_1, \bar{J}_2)$ given by

$$\begin{aligned} \bar{\psi}_1 &= \psi_1 + \delta(\bar{J}_1 + a_2\bar{J}_2), & \bar{\psi}_2 &= \psi_2 + \delta(a_2\bar{J}_1 + a_3\bar{J}_2), \\ \bar{J}_1 &= J_1 + \delta\sin(\psi_1), & \bar{J}_2 &= J_2 + \delta\epsilon\sin(\psi_2). \end{aligned}$$

- T is related to the time-δ map of the flow associated to the Hamiltonian

$$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + \cos\psi_1 + a_2 J_1 J_2 + a_3\frac{J_2^2}{2} + \epsilon\cos(\psi_2),$$

- **4 fixed points:** For $\epsilon d > 0$, $d = a_3 - a_2^2$, $|\epsilon| \ll 1$ and $\delta \lesssim 2$

$p_1 = (0, 0, 0, 0)$ HH, $p_2 = (\pi, 0, 0, 0)$ EH, $p_3 = (0, \pi, 0, 0)$ HE, $p_4 = (\pi, \pi, 0, 0)$ EE.

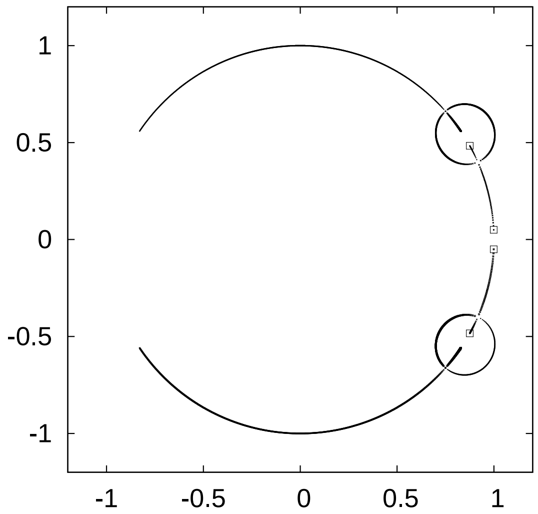
$\rightarrow T$ models the dynamics at a double resonance, it was derived from BNF around an EE point of a symplectic map in V. Gelfreich, C. Simó & AV, *Dynamics of 4D symplectic maps near a double resonance*, Phys D 243(1), 2013.

Motivation: Transition to complex unstable

$\longrightarrow$ If $d>0$ (definite case) the EE point remains EE for all ϵ and δ.
$\longrightarrow$ If $d<0$ (non-definite case) the point p_4 suffers a Krein collision at

$$\epsilon = \left(-(2a_3-4d) \pm \sqrt{(2a_3-4d)^2-4a_3^2} \right) / (2a_3^2),$$

and becomes a **complex-unstable** point (**Hamiltonian-Hopf bifurcation**).



Eigenvalues of $DT(p_4)$ for

$\delta = 0.5$, $a_2 = 0.5$, $a_3 = -0.75$ (hence $d = -1$)

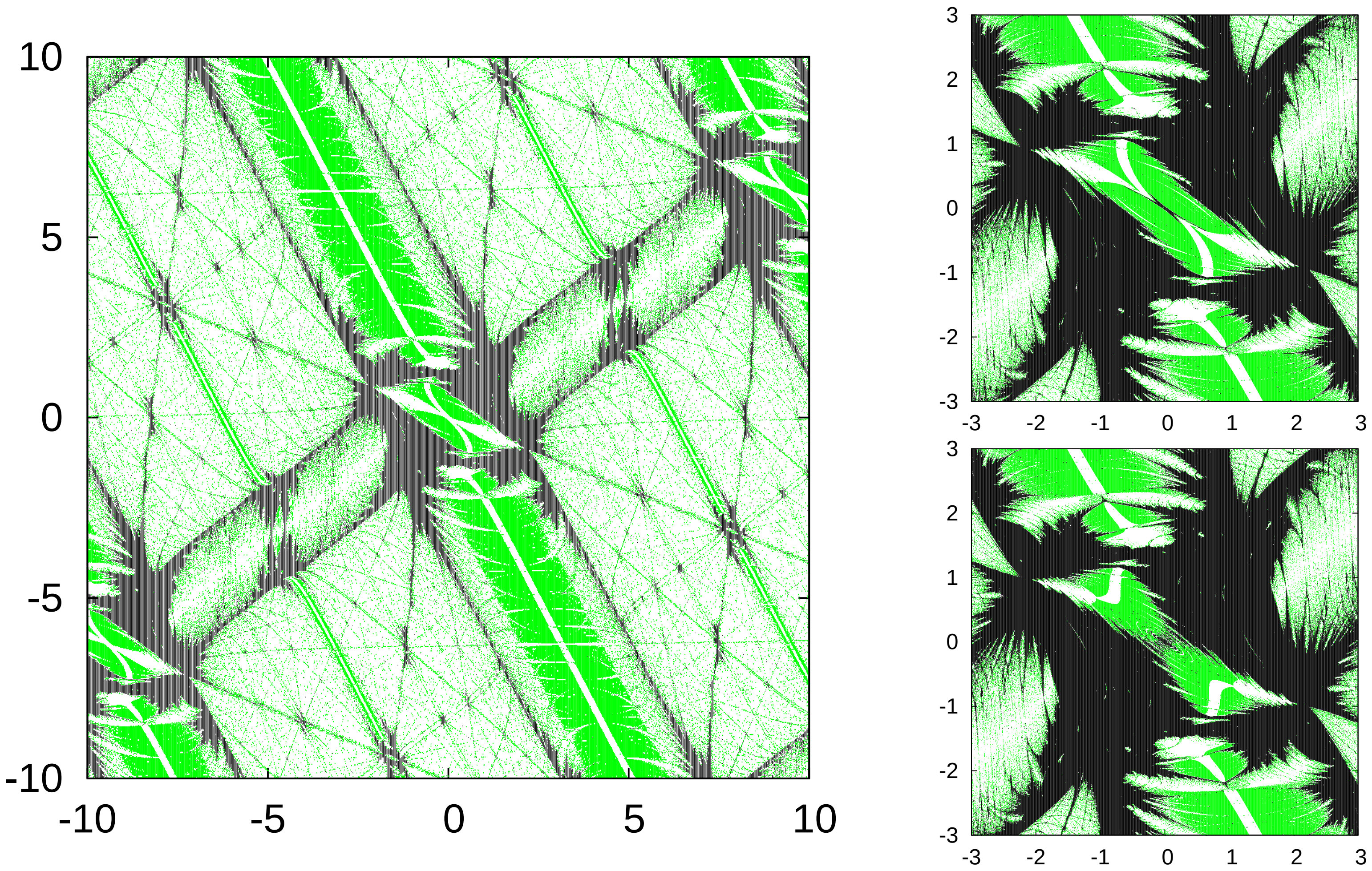
and ϵ from -0.01 (squares) to -20. The (first) Krein collision takes place at $\epsilon^c = -4/9$ at a collision angle $\hat{\theta}_K = \arctan(\sqrt{23}/11)$.

$\longrightarrow$ The CS point has 2D stable/unstable invariant manifolds. $\longrightarrow$ Next plots show their role!

$\longrightarrow$ The previous considerations also hold for H: the eigenvalues collide at the imaginary axis and the 2-dof analogous Hamiltonian-Hopf bifurcation takes place. **Later:** differences discrete/continuous cases in the splitting of the 2D inv. manifolds.

Motivation: Dynamical consequences

Lyapunov exp. MEGNO, i.c. on $\psi_1 = \psi_2 = 0$: white $\rightarrow$ regular, green $\rightarrow$ mild chaos, black $\rightarrow$ chaos.



Left: $\epsilon = -0.4$. Right: top $\epsilon = -0.44$, bottom: $\epsilon = -0.45$. (Rec: $\epsilon^c = -4/9$.)

$\rightarrow$ Lyapunov inv. curves families, local character of the bifurcation, evolution to global connection,...

Goal of this work

We want...

1. Analysis of the Hamiltonian-Hopf bifurcation for 4D maps.
2. Geometry of the 2D invariant manifolds: behaviour of the splitting for the 4D map.

$\longrightarrow$ But, **previously,** we review the 2-d.o.f. analogous Hamiltonian-Hopf case.

1. Sokolskii NF.
2. Splitting of the invariant manifolds: Reduction to a 2D near-the identity area-preserving map.

$\longrightarrow$ **Important:** How are both cases related?

1. Main idea: Takens NF + interpolating Hamiltonian
2. Differences in the behaviour of the splitting: energy function

2-dof Hamiltonian Hopf (HH): Sokolskii NF

2-dof HH codim 1: Consider a 1-param. family of 2-dof Hamiltonians H_ν undergoing a HH bifurcation (at the origin).

Concretely: for $\nu > 0$ elliptic-elliptic, $\nu < 0$ complex-saddle.

Analysis of the HH bifurcation $\to$ Reduction to **Sokolskii NF**:

1. Taylor expansion at $\mathbf{0}$: $H_\nu = \sum_{k\geq 2} \sum_{j\geq 0} \nu^j H_{k,j}$, where $H_{k,j} \in \mathbb{P}_k$ homogeneous of order k.
2. Williamson NF (double purely imaginary eigenvalues):
$H_{2,0} = -\omega(x_2 y_1 - x_1 y_2) + \frac{1}{2}(x_1^2 + x_2^2)$.
3. Use Lie series to order-by-order simplify $H_{2,j}, j > 1$ and $H_{k,j}, k > 2, j > 0$.
But: **non-semisimple** linear part!
Then, at each order (k, j), one looks for $G \in \mathbb{P}_k$ s.t.

$$H_{k,j} + \mathrm{ad}_{H_2}(G) \in \mathrm{Ker}\,\mathrm{ad}_{H_2}^{\top}.$$

2-dof HH: Sokolskii NF

4. Introducing the Sokolskii coordinates ($dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = dR \wedge dr + d\Theta \wedge d\theta$)

$$y_1 = r\cos(\theta),\ y_2 = r\sin(\theta),\ R = (x_1 y_1 + x_2 y_2)/r,\ \Theta = x_2 y_1 - x_1 y_2,$$

one has $H_2^\top = -\omega\Theta + \frac{1}{2}r^2$ and

$$\text{NF}(H_\nu) = -\omega\Gamma_1 + \Gamma_2 + \sum_{\substack{k,l,j \geq 0 \\ k+l \geq 2}} a_{k,l,j}\,\Gamma_1^k\,\Gamma_3^l\,\nu^j.$$

where

$$\Gamma_1 = x_2 y_1 - x_1 y_2,\ \Gamma_2 = (x_1^2 + x_2^2)/2 \text{ and } \Gamma_3 = (y_1^2 + y_2^2)/2.$$

5. Introducing $\nu = -\delta_\nu^2$, and rescaling $x_i = \delta_\nu^2 \tilde{x}_i$, $\omega y_i = \delta_\nu\,\tilde{y}_i$, $i = 1, 2$, $\omega t = \tilde{t}$, one has

$$\boxed{\text{NF}(\tilde{H}_{\delta_\nu}) = -\tilde{\Gamma}_1 + \delta_\nu\left(\tilde{\Gamma}_2 + a\tilde{\Gamma}_3 + \eta\tilde{\Gamma}_3^2\right) + \mathcal{O}(\delta_\nu^2).}$$

The $\tilde{\Gamma}_i$ written in terms of the Sokolskii coordinates are given by

$$\tilde{\Gamma}_1 = \Theta,\ \tilde{\Gamma}_2 = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right),\ \text{and } \tilde{\Gamma}_3 = \frac{r^2}{2}.$$

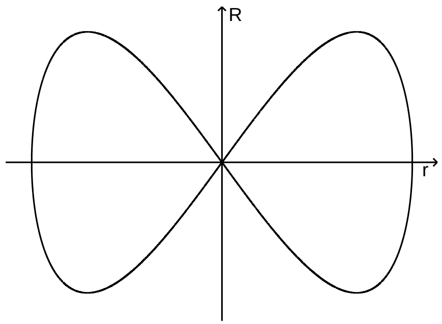
2-dof HH: invariant manifolds

For $\nu < 0$ the origin has stable/unstable inv. manifolds $W^{s/u}(\mathbf{0})$. Note that

- $W^{s/u}(\mathbf{0})$ are contained in the zero energy level of NF$(\tilde{H}_{\delta_\nu})$.
- $\{\tilde{\Gamma}_1, \tilde{\Gamma}_2\} = \{\tilde{\Gamma}_1, \tilde{\Gamma}_3\} = 0 \Rightarrow \tilde{\Gamma}_1$ is a formal first integral of NF$(\tilde{H}_{\delta_\nu})$. Hence $\tilde{\Gamma}_1 = 0$ on $W^{s/u}(\mathbf{0})$.

Then, **ignoring** $\mathcal{O}(\delta_\nu^2)$ terms, $W^{s/u}(\mathbf{0})$ are given by $R^2 + ar^2 + \eta r^4/2 = 0$, which is the zero energy level of a **Duffing Hamiltonian system**.

$\Rightarrow\ W^{u/s}(\mathbf{0})|_{(R,r)\text{-plane}}$ form a **figure-eight** (for $a<0$, $\eta>0$; unbounded otherwise! but only $r>0$ has sense!).



The 2D $W^{s/u}(\mathbf{0})$ are rotated around the origin (on $W^{s/u}(\mathbf{0})$ one has $\Theta=0$, $\dot{\theta}=1$).

For the truncated NF (i.e. ignoring $\mathcal{O}(\delta_\nu^p)$-terms, $p > 1$) the 2D stable/unstable inv. manifolds coincide. **But:** For the complete 2-dof Hamiltonian **they split!**

2-dof HH: splitting of inv. manifolds

The **asymptotic expansion** of this splitting has been obtained in

J.P.Gaivao, V.Gelfreich, *Splitting of separatrices for the Hamiltonian-Hopf bifurcation with the Swift-Hohenberg equation as an example*, Nonlinearity 24(3), 2011.

$$\alpha \sim A\delta_\nu^B \exp\left(\frac{-\pi}{\sqrt{-a\delta_\nu}}\right) \sim A|\text{Re } \lambda|^B \exp\left(\frac{-\pi\,|\text{Im } \lambda|}{|\text{Re } \lambda|}\right)$$

Main idea: The exponential part of this formula can be obtanied by reducing to a near Id family of analytic APMs + Fontich-Simó thm. (upper bounds are generic!).

Consider $\Sigma = \{\theta = 0\}$ (but in Cartesian coord. to avoid singularities) and

$T_{\delta_\nu} : \Sigma \to \Sigma$ (Poincaré map of the full 2-dof Hamiltonian) $\rightsquigarrow$ separatrices split,

$T^0_{\delta_\nu} : \Sigma \to \Sigma$ (Poincaré map of the truncated 2-dof Hamiltonian, ignoring $\mathcal{O}(\delta_\nu^2)$) $\rightsquigarrow$ Homoclinic loop.

Then, $T^0_{\delta_\nu}(R, r, \Theta, \theta) = (\phi^X_{2\pi}, \Theta, \theta \bmod 2\pi)$, being X the vector field

$$\dot{R} = \delta_\nu\left(ar + \eta r^3\right), \quad \dot{r} = -\delta_\nu R, \quad \longleftarrow \text{Duffing!}$$

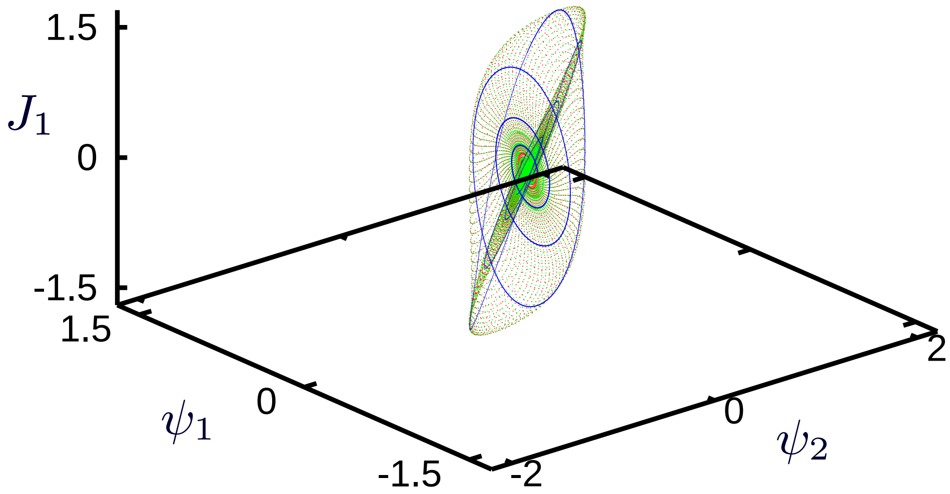
which has a homoclinic solution $\gamma(t)$ with nearest singularity to the real axis $\tau = i\pi/2\sqrt{-a}\delta_\nu$ and dominant eigenvalue $\mu = 2\pi\sqrt{-a\delta_\nu}$ (then rescale time by $\sqrt{-a}\delta_\nu$). But $T_0^{\delta_\nu} = (\hat{T}_0^{\delta_\nu})^2$, being $\hat{T}_0^{\delta_\nu}$ close to -Id $\Rightarrow$ use $\mu/2$ instead of μ in the exponential part of the upper bound $C\exp\left(-2\pi(\text{Im } \tau - \eta)/\mu\right)$.

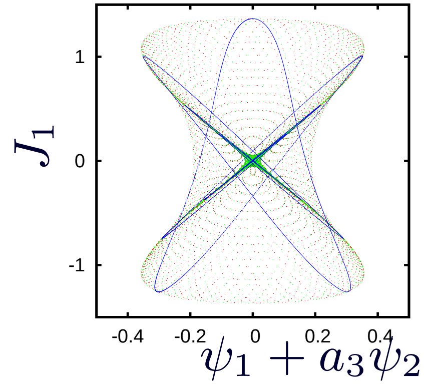
2-dof HH: the example

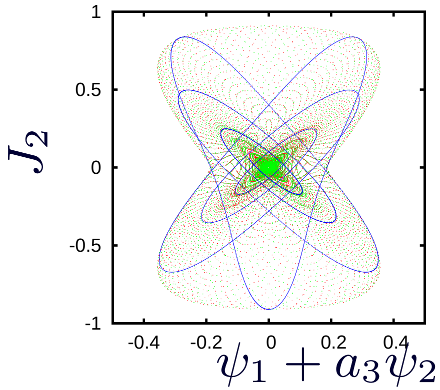
$$H(\psi_1,\psi_2,J_1,J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon\cos(\psi_2)$$

Reversibility: $(\psi_1,\psi_2,J_1,J_2)\in W^u(\mathbf{0})$ then $(-\psi_1,-\psi_2,J_1,J_2)\in W^s(\mathbf{0})$.

This suggests to consider $\Sigma=\{\psi_1=0,\ \psi_2=0\}$ and to look for homoclinic points in Σ.







$a_2=0.5,\ a_3=-0.75,\ \epsilon=-0.5\ (\epsilon^c=-4/9)$

Methodology to get a homoclinic point in Σ (I)

One can locally represent W^u as a series

$$\mathcal{G}(s_1, s_2) = \sum_{i+j\geq 0} a_{i,j} s_1^i s_2^j, \qquad a_{i,j} \in \mathbb{R}$$

where $s_1, s_2 \in \mathbb{R}$ are (real) local parameters in a fundamental domain (an annulus) $\leadsto$ parameterisation method.

Then, one can propagate the local representation and get the invariant manifolds (e.g. using Taylor integrator).

Main steps:

1. Compute the local parameterisation of W^u (order by order).
2. Truncate it to order N and look for r_* (radius in (s_1, s_2)) such that the invariance equation is verified up to a given tolerance `tol`. The points on the circle of radius r_* can be parameterised by an angle θ.

Methodology to get a homoclinic point in Σ (II)

3. To compute θ s.t. parameterises a point on Σ we proceed as follows:

   (a) Discretize θ: $\{\theta_i\}_{i=1,\ldots,1000}$.

   Each θ_i gives an initial condition $\longrightarrow$ integrate (Taylor method).

   (b) Integrate each i.c. up to $\{\psi_2 = 0\}$.

   Problem: $\{\psi_2 = 0\}$ is crossed many times before we arrive to Σ!!

   We proceed as follows:

   i. We fix a number `m` and we integrate up to the `m` crossing with $\psi_2 = 0$. Hence, for each i we obtain a point on $\psi_2 = 0$. Denote by $\psi_{1,i}$ the corresponding coordinate of this point.

   ii. If for a concrete i one has $\psi_{1,i}\psi_{1,i-1} < 0$ then we look for $\theta \in (\theta_i - 1, \theta_i)$ such that $\psi_1 = 0$ in Σ (e.g. secant method). Otherwise, if there is not i verifying this last condition, we increase `m`.

$\Longrightarrow$ We get a homoclinic point on Σ (first intersection!).

2-dof HH: Computing the splitting

Using the last methodology one obtains (s_1^h, s_2^h) corresponding a homoclinic point p_h on Σ (at the first intersection!). $\rightsquigarrow$ The homoclinic orbit was shown in the last plot!

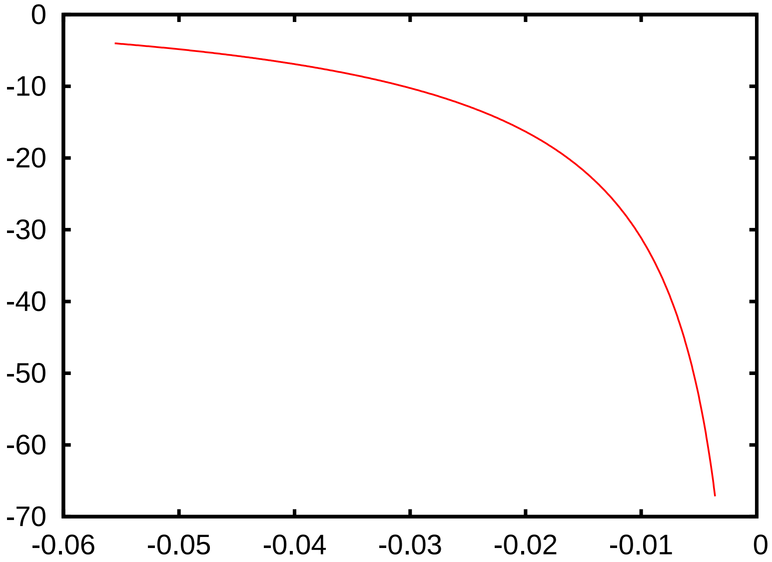
To measure the splitting angle α at p_h:

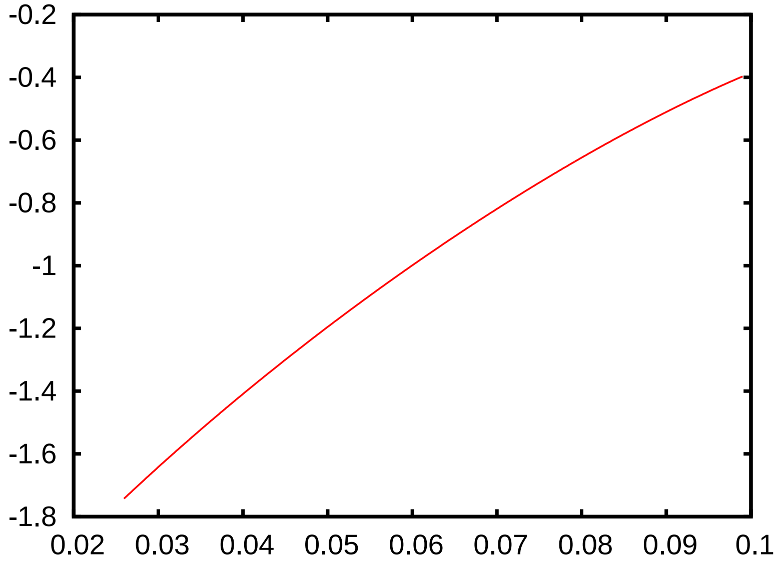
1. **Compute a basis of** $T_{X_h^0}(W^u_{loc}(\mathbf{0})) \rightsquigarrow v_t^0 = \frac{\partial \mathcal{G}}{\partial s_1}(s_1^h, s_2^h)$,$v^0_{\mathtt{vf}}$
2. **Transport the vectors to** $\Sigma \rightsquigarrow v_t^\Sigma$,$v^\Sigma_{\mathtt{vf}}$ (integrating variational eqs.)
   These vectors form a basis of $T_{p_h}(W^u(\mathbf{0}))$.
3. **Compute an orthogonal basis of** $T_{p_h}(W^u(\mathbf{0})) \rightsquigarrow w_1$,$v^\Sigma_{\mathtt{vf}}$
4. **Compute the splitting angle.** By reversibility, from $w_1 \in T_{p_h}(W^u(\mathbf{0}))$ we obtain a vector $w_2 \in T_{p_h}(W^s(\mathbf{0}))$. Then,

$$\boxed{\alpha = \text{angle}(w_1, w_2)}$$

2-dof HH: Checking the behaviour of α

$$H(\psi_1,\psi_2,J_1,J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon\cos(\psi_2)$$





Left: $\log(\alpha)$ vs. $\epsilon - \epsilon^c$. Right: Re$(\lambda)\log(\alpha)$ vs. Re(λ).

Recall: $\alpha \sim \tilde{A}(|\text{Re }\lambda|)^B \exp\left(\frac{C}{|\text{Re }\lambda|}\right)$, where $C = -\pi\,|\text{Im }\lambda|$.

For $a_2 = 0.5$, $a_3 = -0.75$ one gets $C = \sqrt{2}\pi/3 + \mathcal{O}(\nu)$ (Sokolskii NF).

Fitting function (right plot): $f(x) = A\,x + B\,x\,\log(x) + C$.

$\rightsquigarrow$ It perfectly fits the behaviour!

Up to this point: **2-dof Hamiltonian-Hopf** bifurcation.

1. Everything was "more or less" well-known: Sokolskii NF, geometry of the invariant manifolds, the splitting α,...
2. α behaves as expected for a near-the-identity family of 2D APM.

Guiding example: $H = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2)$.

Now: 4D discrete Hamiltonian-Hopf!

Guiding example: the 4D symplectic map T given by

$$\begin{aligned} \bar{\psi}_1 &= \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2), & \bar{\psi}_2 &= \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2), \\ \bar{J}_1 &= J_1 + \delta \sin(\psi_1), & \bar{J}_2 &= J_2 + \delta\epsilon \sin(\psi_2). \end{aligned}$$

The origin undergoes a HH bif. and 2D stable/unstable manifolds are born.

Question: Behaviour of the splitting of the 2D inv. manifolds?

Planning:

First: Numerical exploration of T.

$\longrightarrow$ Computation of the invariant manifolds.

$\longrightarrow$ Behaviour of the splitting.

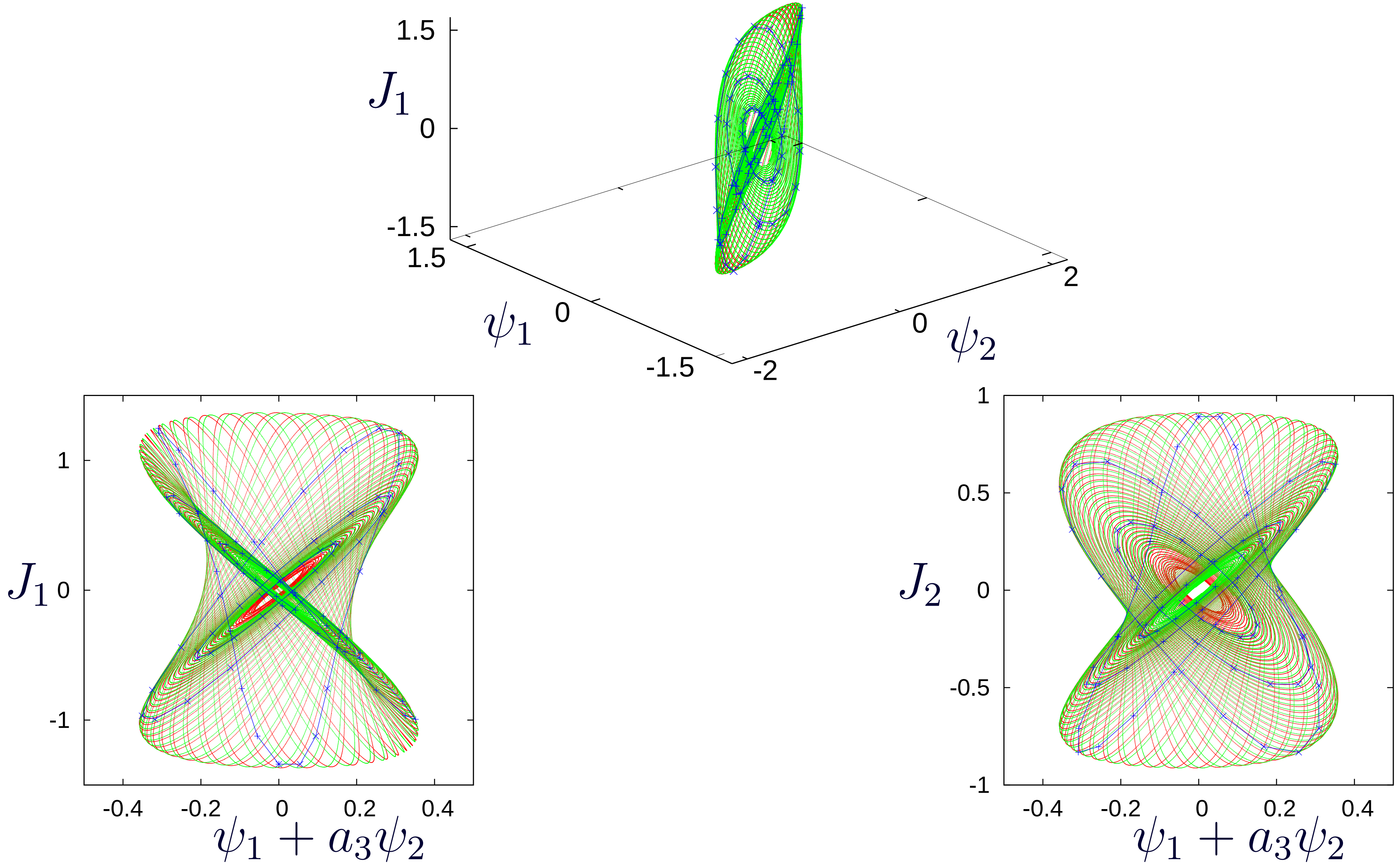
$\longrightarrow$ A naive justification of the behaviour observed.

After: General theoretical results on splitting of inv. manifolds for the 4D HH.

$\longrightarrow$ Upper bounds from a suitable energy function.

T: Invariant manifolds

One can compute $W^{u/s}(\mathbf{0})$ and $p_h \in \Sigma = \{\psi_1 = \psi_2 = 0\}$ (similarly to the 2-dof case).



T: Splitting volume V

We compute the volume of a $4D$ parallelotope defined by two pairs of vectors tangent to W^u and W^s at $p_h \in \Sigma$:

$G(s_1, s_2)$ - local parameterisation

(s_1^h, s_2^h) - local parameters s.t. $T^N(s_1^h, s_2^h) = p_h$, $N > 0$.

1. Consider the vectors:

$$\tilde{v}_1 = (\partial G/\partial s_1)(s_1^h, s_2^h), \quad \tilde{v}_2 = (\partial G/\partial s_2)(s_1^h, s_2^h) \quad \leftarrow \text{ tangent to } W^u(\mathbf{0})$$

2. Transport these vectors under T to p_h and consider, by the reversibility,

$$\tilde{v}_3 = R(\tilde{v}_1^{p_h}), \quad \tilde{v}_4 = R(\tilde{v}_2^{p_h}) \quad \leftarrow \text{ tangent to } W^s(\mathbf{0})$$

3. Finally, normalize them $v_j = \tilde{v}_j^{p_h}/\|\tilde{v}_j^{p_h}\|$, $j = 1, \ldots, 4$ and define

$$\boxed{V = \det(v_1, v_2, v_3, v_4)}$$

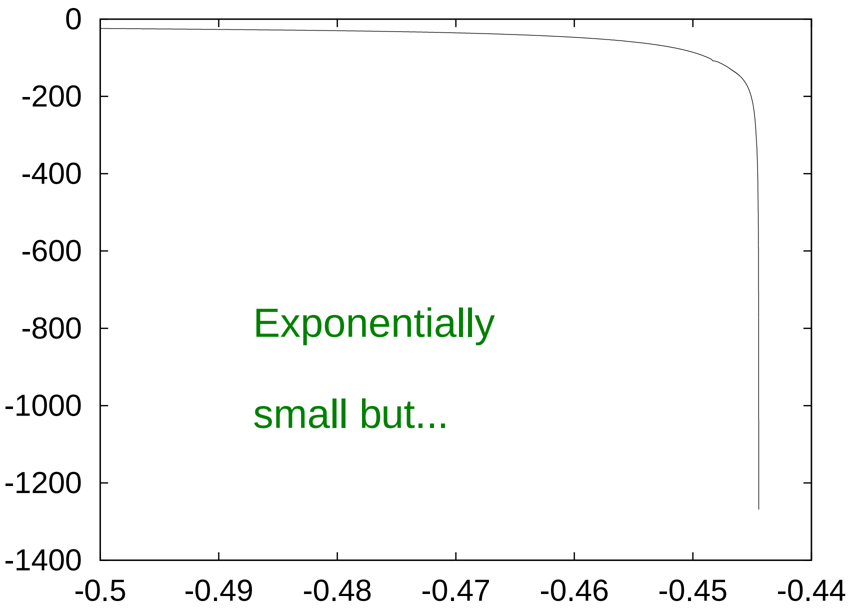
Question: How does V behave as $\epsilon \to \epsilon^c$?

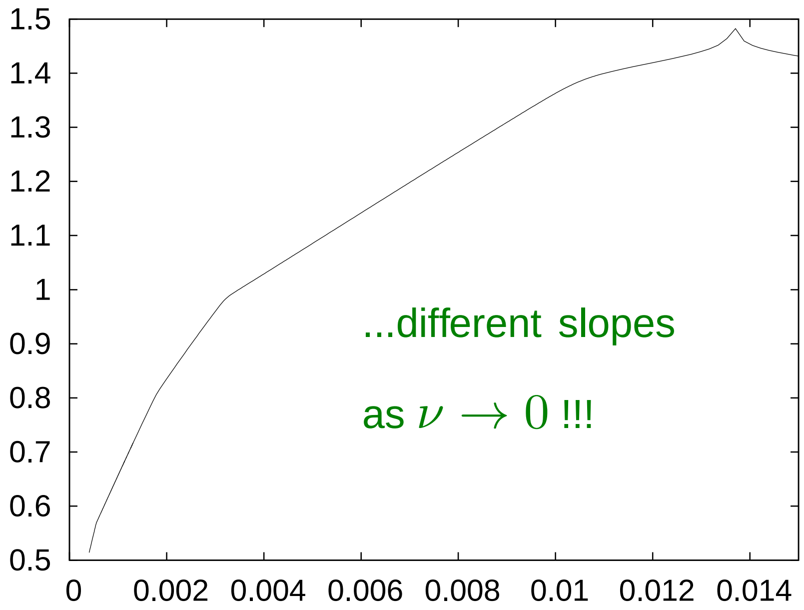
T: Behaviour of V

$$T: \quad \bar\psi_1 = \psi_1 + \delta(\bar J_1 + a_2 \bar J_2), \quad \bar\psi_2 = \psi_2 + \delta(a_2 \bar J_1 + a_3 \bar J_2),$$
$$\bar J_1 = J_1 + \delta \sin(\psi_1), \quad \bar J_2 = J_2 + \delta\epsilon \sin(\psi_2).$$

Fixed a_2, a_3 one has $\epsilon^c = \epsilon^c(a_2, a_3)$. The (Krein) collision angle $\hat\theta_{\mathrm{K}}$ depends on δ.

$a_2 = 0.5$, $a_3 = -0.75 \rightsquigarrow \epsilon^c = -4/9$. $\quad \delta = 0.5 \rightsquigarrow \theta_{\mathrm{K}} = \arctan(\sqrt{23}/11)/2\pi$ ("$\in \mathbb{R}\setminus\mathbb{Q}$").





Left: $\log V$ vs. ϵ. Right: $h|\log(V)|$ vs. h $(h = \log(\lambda))$.

Naive explanation of the behaviour of V

Consider a (generic) symplectic map F in $\mathbb{R}^4$ undergoing a HH bif.

Discrete HH bif. $\rightsquigarrow$ codim 2 bif $\rightsquigarrow$ Let δ_t, ϵ_t be the unfolding parameters.

δ_t: Collision angle $\hat{\theta}_{\mathrm{K}} = 2\pi(q/m + \delta_t)$.

ϵ_t: Measures the relative distance to the critical parameter.

Different (naive) important aspects:

1. "Two" exp. small effects: one within the Hamiltonian itself (already studied!), the other measures the "map-Hamiltonian distance".
2. "Two" frequencies: "Duffing" and its $2\pi\theta_{\mathrm{K}}$-perturb. + **"time" frequency**.
3. The Hamiltonian part is known $\Rightarrow$ only necessary to measure the second effect. **But:** We have a "privilegiated direction" (the time!) $\Rightarrow$ we will use an **energy function** to measure the splitting in that direction (instead of using the splitting potential or the Melnikov vector which measures both effects together).

Towards a sharp upper bound of the splitting (I)

Idea: It is enough to measure the “Hamiltonian-map distance”.

Let F_{ϵ_t} be a family of symplectic maps s.t. at $\epsilon_t = 0$ undergoes a HH bifurcation. The inv. manifolds $W^{u/s}(\mathbf{0})$ are given by $u(\alpha, t)$ and $v(\alpha, t)$ resp., where $(\alpha, t) \in [t_0, t_0 + h) \times S^1$. This defines FD's $\mathcal{D}^{u/s}$.

Main result: Assume that

(H1) There exists an energy function E, i.e. such that $E \circ F_{\epsilon_t} = E$, defined in a neighbourhood of the fundamental domain $\mathcal{D}^s$ such that $E(v(\alpha, t)) = 0$. Moreover we assume that E and $v(\alpha, t)$ can be analytically extended to a neighbourhood of $W^u(\mathbf{0})$ within $\mathcal{D}^u$ (by iteration of $F_{\epsilon_t}^{-1}$).

We define the *splitting function*:

$$\psi(\alpha, t) = E(u(\alpha, t))$$

Towards a sharp upper bound (II)

(H2) There is a (limit) vector field

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^2,$$

such that f is analytic, it possesses a hyperbolic saddle fixed point and a homoclinic orbit $\sigma(t)$ associated to it, and satisfies that compact pieces of the real invariant manifolds of F_{ϵ_t} are ϵ_t-close to an embedding of $\mathbb{S}^1 \times \{\sigma(t),\ t \in \mathbb{R}\}$ for $\epsilon_t > 0$ small enough.

(H3) F_{ϵ_t} can be extended analytically to a neighbourhood of

$$\{\alpha \in \mathbb{C}/2\pi\mathbb{Z}, |\text{Im}\,\alpha| < \rho\} \times \{\sigma(t), |\text{Im}\,t| < \tau\}$$

for some $0 < \tau < \tau_0$ and $0 < \rho < \rho_0$.

Towards a sharp upper bound (Result)

Under (H1), (H2) and (H3)...

(i) **Rational Krein collision.** Let $\theta_{\mathrm{K}} = p/q$, with $(p, q) = 1$. Then, there exists $\epsilon_t^0 > 0$ s.t. for $\epsilon_t < \epsilon_t^0$

$$|\psi(\alpha, t)| \leq K \exp(-C/h), \qquad C, K > 0.$$

(ii) **Irrational Krein collision.** Let $\theta_{\mathrm{K}} \in \mathbb{R} \setminus \mathbb{Q}$. Then, ψ is bounded by a function that is exponentially small in a parameter γ, s.t. $\gamma \searrow 0$ when $h \searrow 0$. Moreover, the dominant harmonic $k(h)$ of ψ changes infinitely many times as $h \to 0$.

Idea: Bounding the Fourier coefficients of ψ, one gets

$$|\psi(\alpha, t)| < K \sum_{(k,n)\in\mathbb{Z}_*^2} \exp(\underbrace{-2\pi|n - \theta_0 k|\tau/h - |k|\rho}_{\beta_k}).$$

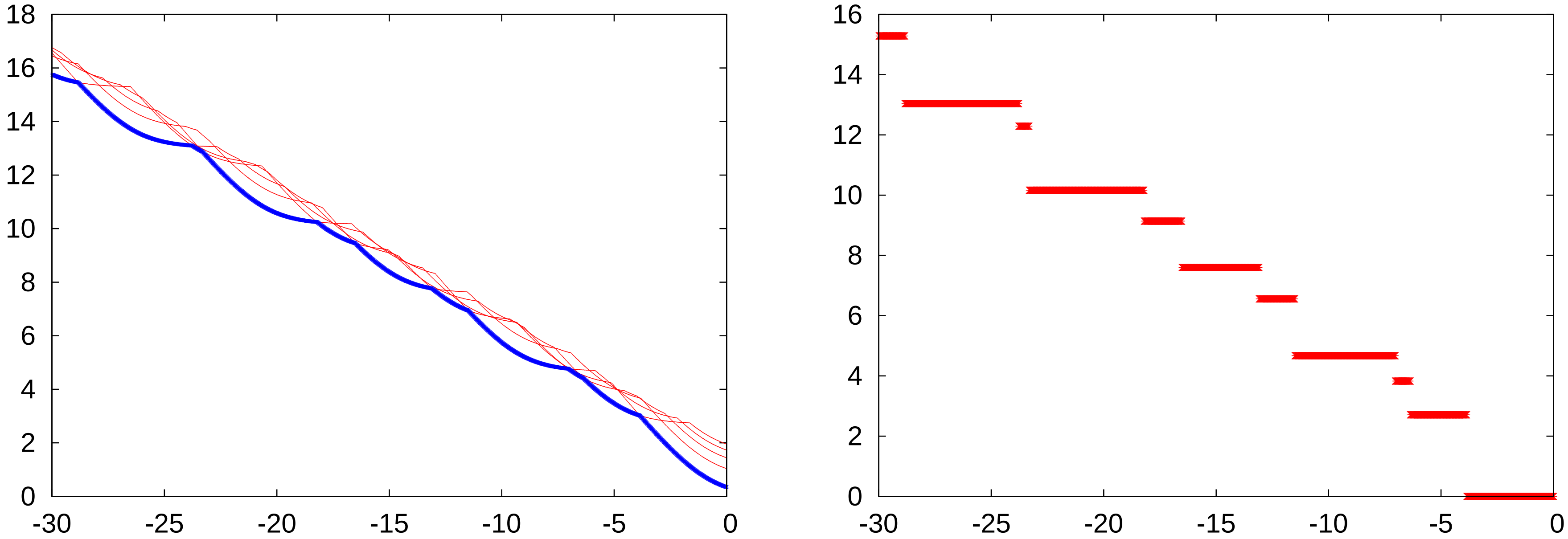
Then we look for $k = k_*(h) > 0$ s.t. the dominant coefficient β_{k_*} in the exponential bound is minimum (different cases according to the properties of θ_{K}).

Map T: fit of the volume V (I)

$$T: \quad \bar{\psi}_1 = \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2), \quad \bar{\psi}_2 = \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2),$$
$$\bar{J}_1 = J_1 + \delta \sin(\psi_1), \quad \bar{J}_2 = J_2 + \delta\epsilon \sin(\psi_2).$$

$a_2 = 0.5$, $a_3 = -0.75 \rightsquigarrow \epsilon^c = -4/9$.

We look for the dominant coefficients $\beta_{k(h)}$. They depend on θ_K and $h = \log(\lambda) = \mathcal{O}(\sqrt{|\epsilon - \epsilon^c|})$. We fix $\theta_K = \arctan(\sqrt{23}/11)/2\pi$.



Left: first five dominant exponents β_k as a function of h. Right: values of k_* corresponding to the minimum exponent β_k. Both in $\log - \log$ scale.

Map T: fit of the volume V (II)

- We have $k_* = 1, 15, 46, 107, 703, 2002, 9307, 25919,$ ... as $h \to 0$.
- The values of k_* are related to the approximants of $\theta_{\mathtt{K}} \approx 0.0654346\,2308$:
  $1/15,\ 3/46,\ 4/61,\ 7/107,\ 39/596,\ 46/703,\ 85/1299,\ 131/2002, \ldots$
- Not all the approximants produce a change of $k_*(h)$ as $h \to 0$, only those that are smaller than $\theta_{\mathtt{K}}$ play a role (except the first one $1/15 > \theta_0$).
- The lenght of the interval in h where $k_*(h)$ dominates depends on the CFE$(\theta_{\mathtt{K}}) = [15,\ 3,\ 1,\ 1,\ 5,\ 1,\ 1,\ 3,\ 1,\ 2, \ldots]$, but also on the constants in front of the exponential terms of V (terms with larger β_k can dominate for finite $h > 0$!!!)

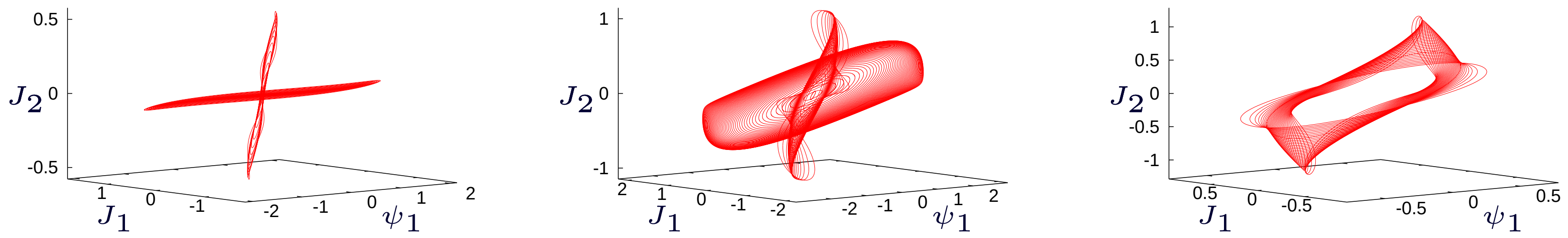
Conclusion: The numerical fit data show that the different slopes observed are related to the different values $k_*(h)$ obtained $\rightsquigarrow$ OK!!!

Final comments I

1. **Other aspects** related to the HH bifurcation for 4D maps have been also investigated (preprint).

   For example:

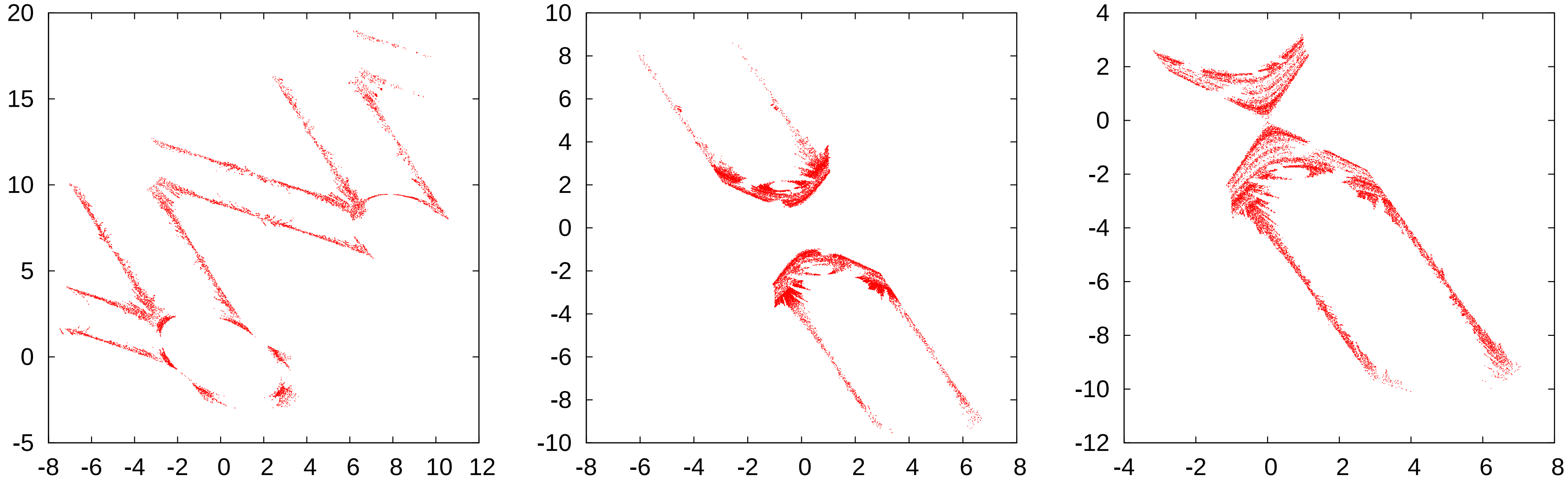
   (a) Structure of the Lyapunov families of invariant curves (analytic results on: the detachment of the Lyapunov families, analysis of the rational and irrational collision angle θ_K cases, stability of the inv. curves, ...).



Detachment of the Lyapunov families of invariant curves for T:
$\epsilon = -0.1, -0.4$ and -0.5 ($\epsilon^c = -4/9$).

Final comments II

(b) Possible diffusive patterns **through and around** the double resonance.



Left: Positive definite case ($\delta = \epsilon = a_2 = 0.5$ and $a_3 = 1.25$).

Centre/Right: Non-definite case ($\delta = \epsilon = a_2 = 0.5$ and $a_3 = -0.75$).

2. **Many open questions:** Theorem of splitting for a family of 4D maps? Separatrix return map? Diffusive properties (quantitative data)?

...but this is left for future works...

Thanks for your attention!!