Periodic forcing of a 2-dof Hamiltonian undergoing a Hamiltonian-Hopf bifurcation

Dynamics, Bifurcations and Strange Attractors

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A. Vieiro

Ongoing work with E. Fontich and C. Simó

University of Barcelona (UB)

Goal of this talk

To show some aspects related to the Hamiltonian-Hopf bifurcation in different contexts. The major interest will be on the behaviour of the splitting of the invariant manifolds for

1. 2-dof Hamiltonian system $\longleftarrow$ well known
2. 4D symplectic maps
3. 2-dof Hamiltonian + periodic forcing

Let me start with:

1. An overview of the Hamiltonian-Hopf bifurcation for 2-dof Hamiltonian systems

2-dof Hamiltonian Hopf (HH): Sokolskii NF

2-dof HH codim 1: Consider a 1-param. family of 2-dof Hamiltonians H_ν undergoing a HH bifurcation (at the origin).

Concretely: for $\nu > 0$ elliptic-elliptic, $\nu < 0$ complex-saddle.

Analysis of the HH bifurcation $\to$ Reduction to **Sokolskii NF**:

1. Taylor expansion at $\mathbf{0}$: $H_\nu = \sum_{k\geq 2}\sum_{j\geq 0} \nu^j H_{k,j}$, where $H_{k,j} \in \mathbb{P}_k$ homogeneous polynomial of order k.
2. Williamson NF (double purely imaginary eigenvalues $\pm i\omega$): $H_{2,0} = -\omega(x_2 y_1 - x_1 y_2) + \frac{1}{2}(x_1^2 + x_2^2)$.
3. Use Lie series to order-by-order simplify $H_{2,j}, j > 1$ and $H_{k,j}, k > 2, j > 0$.

   But: **non-semisimple** linear part!

   Then, at each order (k, j), one looks for $G \in \mathbb{P}_k$ s.t.

$$H_{k,j} + \mathrm{ad}_{H_2}(G) \in \mathrm{Ker}\,\mathrm{ad}_{H_2}^{\top}.$$

2-dof HH: Sokolskii NF

4. Introducing the Sokolskii coordinates $(dx_1\wedge dy_1+dx_2\wedge dy_2=dR\wedge dr+d\Theta\wedge d\theta)$

$$y_1=r\cos(\theta),\ y_2=r\sin(\theta),\ R=(x_1y_1+x_2y_2)/r,\ \Theta=x_2y_1-x_1y_2,$$

one has $H_2^\top=-\omega\Theta+\frac{1}{2}r^2$ and

$$\mathsf{NF}(H_\nu)=-\omega\Gamma_1+\Gamma_2+\sum_{\substack{k,l,j\geq 0\\ k+l\geq 2}} a_{k,l,j}\,\Gamma_1^k\,\Gamma_3^l\,\nu^j,$$

where

$$\Gamma_1=x_2y_1-x_1y_2,\ \Gamma_2=(x_1^2+x_2^2)/2 \text{ and } \Gamma_3=(y_1^2+y_2^2)/2.$$

5. Introducing $\nu=-\delta_\nu^2$, and rescaling $x_i=\delta_\nu^2\tilde{x}_i, \omega y_i=\delta_\nu\,\tilde{y}_i, i=1,2,$ $\omega t=\tilde{t}$, one has

$$\boxed{\mathsf{NF}(\tilde{H}_{\delta_\nu})=-\tilde{\Gamma}_1+\delta_\nu\left(\tilde{\Gamma}_2+a\tilde{\Gamma}_3+\eta\tilde{\Gamma}_3^2\right)+\mathcal{O}(\delta_\nu^2).}$$

The $\tilde{\Gamma}_i$ written in terms of the Sokolskii coordinates are given by

$$\tilde{\Gamma}_1=\tilde{\Theta},\ \tilde{\Gamma}_2=\frac{1}{2}\left(\tilde{R}^2+\frac{\tilde{\Theta}^2}{\tilde{r}^2}\right),\text{ and } \tilde{\Gamma}_3=\frac{\tilde{r}^2}{2}.$$

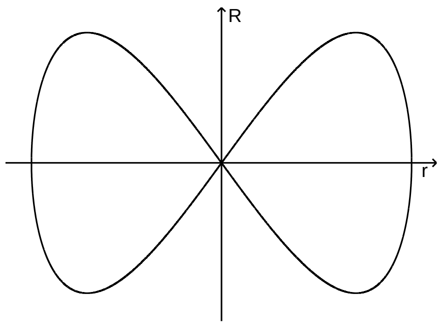
2-dof HH: invariant manifolds

For $\nu < 0$ the origin has stable/unstable inv. manifolds $W^{s/u}(\mathbf{0})$. Note that

- $W^{s/u}(\mathbf{0})$ are contained in the zero energy level of NF$(\tilde{H}_{\delta_\nu})$.
- $\{\tilde{\Gamma}_1, \tilde{\Gamma}_2\} = \{\tilde{\Gamma}_1, \tilde{\Gamma}_3\} = 0 \Rightarrow \tilde{\Gamma}_1$ is a formal first integral of NF$(\tilde{H}_{\delta_\nu})$. Hence $\tilde{\Gamma}_1 = 0$ on $W^{s/u}(\mathbf{0})$.

Then, **ignoring** $\mathcal{O}(\delta_\nu^2)$ terms, $W^{s/u}(\mathbf{0})$ are given by $R^2 + ar^2 + \eta r^4/2 = 0$, which is the zero energy level of a **Duffing Hamiltonian system**.

$\Rightarrow \; W^{u/s}(\mathbf{0})|_{(R,r)\text{-plane}}$ form a **figure-eight** (for $a < 0$, $\eta > 0$; unbounded otherwise! but only $r > 0$ has sense!).



The 2D $W^{s/u}(\mathbf{0})$ are obtained by rotating the right hand side of the figure around the R axis (on $W^{s/u}(\mathbf{0})$ one has $\Theta = 0$, $\dot{\theta} = 1$).

2-dof HH: splitting of inv. manifolds

For the truncated NF (i.e. ignoring $\mathcal{O}(\delta_\nu^p)$-terms, $p > 1$) the 2D stable/unstable inv. manifolds coincide. **But:** For the complete 2-dof Hamiltonian **they split!**

1. Consider $\Sigma = \{\theta = 0\}$. The Poincaré map (in Cartesian coord. to avoid singularities) defines a *near-the-Id family of analytic APMs.*
2. The limit vector field is $\dot{R} = \delta_\nu \left(ar + \eta r^3\right), \ \dot{r} = -\delta_\nu R,$ $\longleftarrow$ Duffing!

   The homoclinic solution $\gamma(t)$ with nearest singularity to the real axis $\tau = i\pi/2\sqrt{-a}\delta_\nu$ and dominant eigenvalue $\mu = 2\pi\sqrt{-a}\delta_\nu$ (then rescale time by $\sqrt{-a}\delta_\nu$).
3. From Fontich-Simó theorem (upper bounds are generic!) it follows

$$\alpha \sim A\delta_\nu^B \exp\left(\frac{-\pi}{\sqrt{-a}\delta_\nu}\right) \sim A|\text{Re }\lambda|^B \exp\left(\frac{-\pi\,|\text{Im }\lambda|}{|\text{Re }\lambda|}\right)$$

The **asymptotic expansion** of this splitting has been obtained in

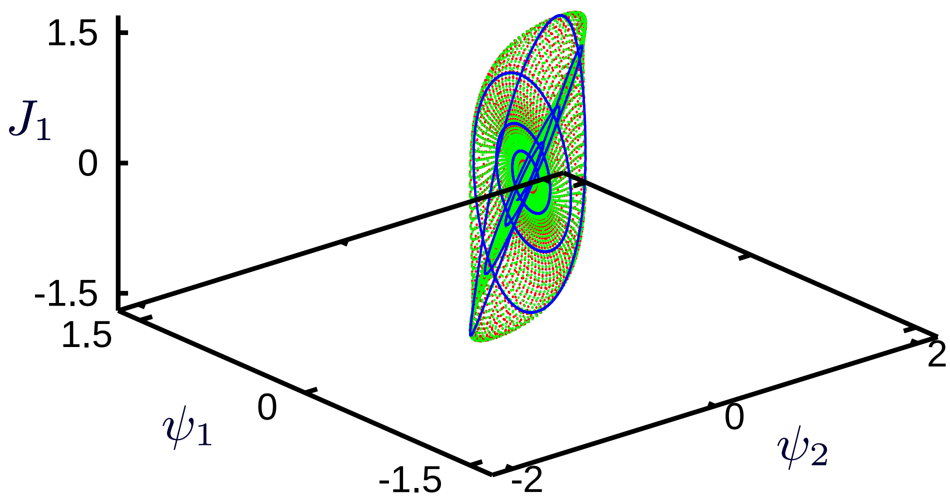
J.P.Gaivao, V.Gelfreich, *Splitting of separatrices for the Hamiltonian-Hopf bifurcation with the Swift-Hohenberg equation as an example*, Nonlinearity 24(3), 2011.

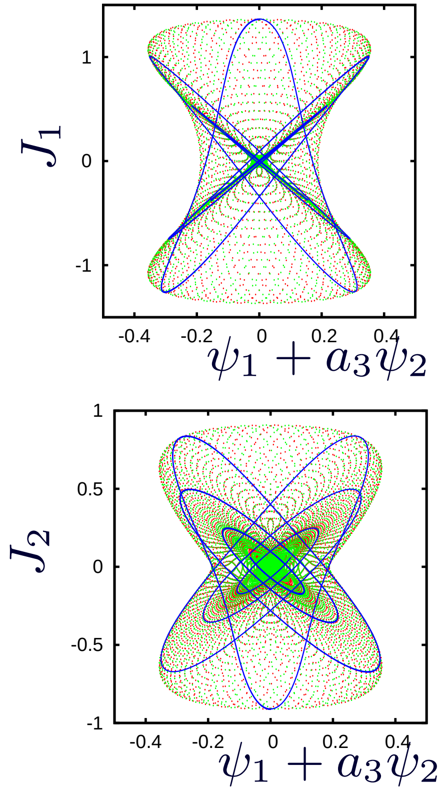
2-dof HH: An example

$$H(\psi_1,\psi_2,J_1,J_2)=\frac{J_1^2}{2}+a_2J_1J_2+a_3\frac{J_2^2}{2}-\cos(\psi_1)-\epsilon\cos(\psi_2)$$

Reversibility: $(\psi_1,\psi_2,J_1,J_2)\in W^u(\mathbf{0})$ then $(-\psi_1,-\psi_2,J_1,J_2)\in W^s(\mathbf{0})$.

This suggests to consider $\Sigma=\{\psi_1=0,\,\psi_2=0\}$ and to look for homoclinic points in Σ.

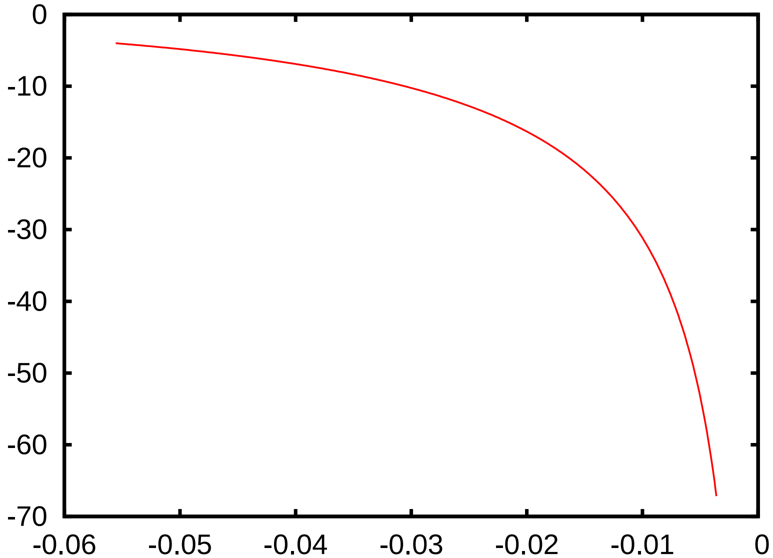




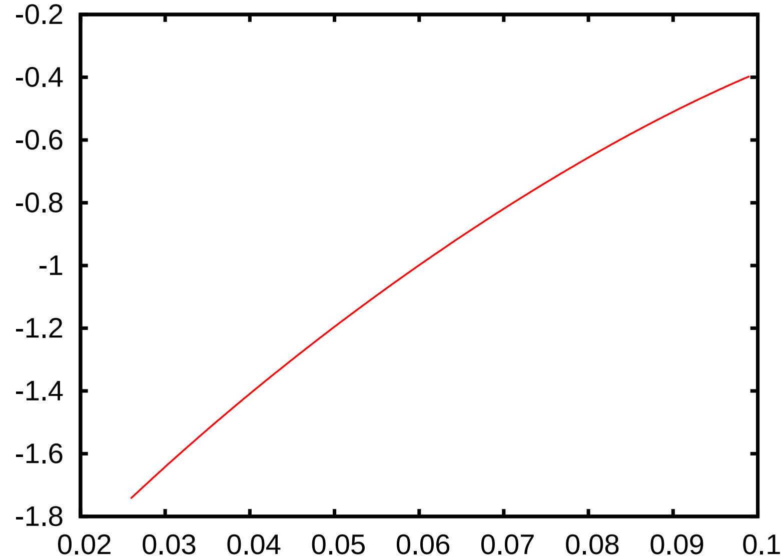
$a_2{=}0.5,\ a_3{=}-0.75,\ \epsilon{=}-0.5\ (\epsilon^c{=}-4/9)$

2-dof HH: Checking the behaviour of α

$$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2)$$



$\log(\alpha)$ vs. $\epsilon - \epsilon^c$



Re$(\lambda) \log(\alpha)$ vs. Re(λ)

Recall: $\alpha \sim \tilde{A}(|\text{Re } \lambda|)^B \exp\left(\frac{C}{|\text{Re } \lambda|}\right)$, where $C = -\pi \, |\text{Im } \lambda|$.

For $a_2 = 0.5$, $a_3 = -0.75$ one gets $C = \sqrt{2}\pi/3 + \mathcal{O}(\nu)$.

Fitting function (right plot): $f(x) = A\,x + B\,x\,\log(x) + C$.

$\rightsquigarrow$ It perfectly fits the behaviour!

Part 2

Up to this point: **2-dof Hamiltonian-Hopf** bifurcation.

1. Everything was “more or less” well-known: Sokolskii NF, geometry of the invariant manifolds, the splitting α,...
2. α behaves as expected for a near-the-identity family of 2D APM.

Let me now continue with:

2. A brief excursion to 4D symplectic maps undergoing a HH bifurcation

A paradigmatic Froeschlé-like map

Consider the map $T : (\psi_1, \psi_2, J_1, J_2) \mapsto (\bar{\psi}_1, \bar{\psi}_2, \bar{J}_1, \bar{J}_2)$ given by

$$\begin{aligned} \bar{\psi}_1 &= \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2), & \bar{\psi}_2 &= \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2), \\ \bar{J}_1 &= J_1 + \delta \sin(\psi_1), & \bar{J}_2 &= J_2 + \delta\epsilon \sin(\psi_2). \end{aligned}$$

- T is related to the time-δ map of the flow associated to the Hamiltonian

$$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + \cos\psi_1 + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} + \epsilon \cos(\psi_2),$$

- **4 fixed points:** For $\epsilon d > 0$, $d = a_3 - a_2^2$, $|\epsilon| \ll 1$ and $\delta \lesssim 2$

  $p_1 = (0, 0, 0, 0)$ HH, $p_2 = (\pi, 0, 0, 0)$ EH, $p_3 = (0, \pi, 0, 0)$ HE, $p_4 = (\pi, \pi, 0, 0)$ EE.

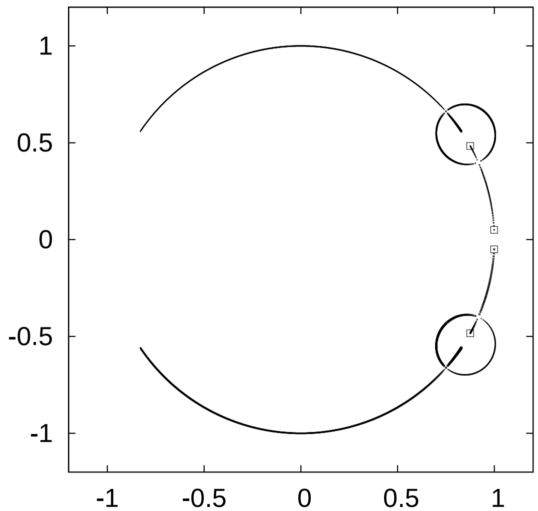
$\rightarrow T$ models the dynamics at a double resonance, it was derived from BNF around an EE point of a symplectic map in V. Gelfreich, C. Simó & AV, *Dynamics of 4D symplectic maps near a double resonance*, Phys D 243(1), 2013.

Motivation: Transition to complex unstable

$\longrightarrow$ If $d > 0$ (definite case) the EE point remains EE for all ϵ and δ.
$\longrightarrow$ If $d < 0$ (non-definite case) the point p_4 suffers a Krein collision at

$$\epsilon = \left(-(2a_3 - 4d) \pm \sqrt{(2a_3 - 4d)^2 - 4a_3^2} \right) / (2a_3^2),$$

and becomes a **complex-unstable** point (**Hamiltonian-Hopf bifurcation**).



Eigenvalues of $DT(p_4)$ for

$\delta = 0.5$, $a_2 = 0.5$, $a_3 = -0.75$ (hence $d = -1$)

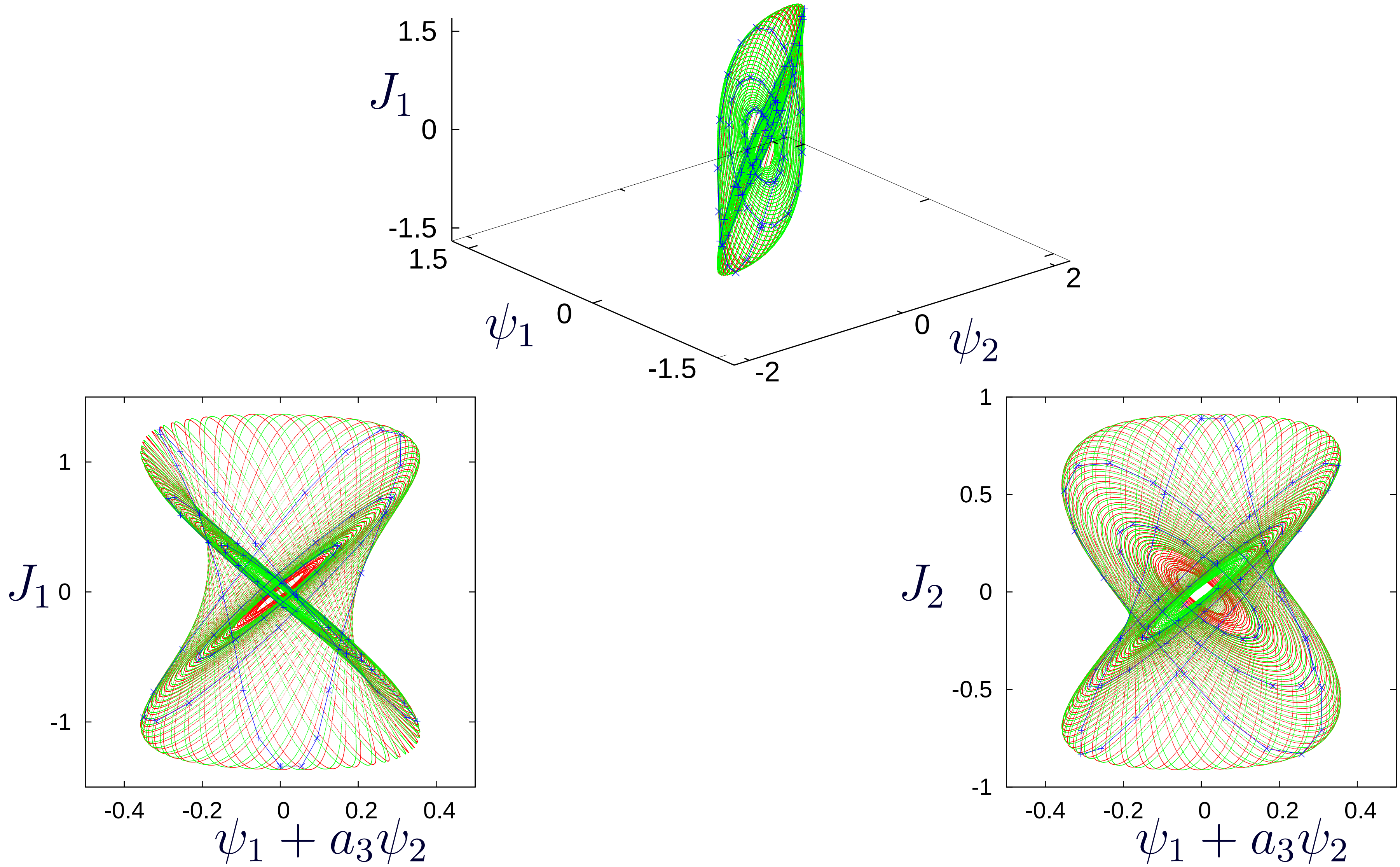
and ϵ from -0.01 (squares) to -20. The (first) Krein collision takes place at $\epsilon^c = -4/9$ at a collision angle $\hat{\theta}_K = \arctan(\sqrt{23}/11)$.

$\longrightarrow$ The CS point has 2D stable/unstable invariant manifolds.

T: Invariant manifolds

One can compute $W^{u/s}(\mathbf{0})$ and a homoclinic point $p_h \in \Sigma = \{\psi_1 = \psi_2 = 0\}$

(similarly to the 2-dof case).



T: Splitting volume V

We compute the volume of a $4D$ parallelotope defined by two pairs of vectors tangent to W^u and W^s at $p_h \in \Sigma$:

$G(s_1, s_2)$ - local parameterisation

(s_1^h, s_2^h) - local parameters s.t. $T^N(s_1^h, s_2^h) = p_h$, $N > 0$.

1. Consider the vectors:

$$\tilde{v}_1 = (\partial G/\partial s_1)(s_1^h, s_2^h), \quad \tilde{v}_2 = (\partial G/\partial s_2)(s_1^h, s_2^h) \quad \leftarrow \text{tangent to } W^u(\mathbf{0})$$

2. Transport these vectors under T^N to p_h and consider, by the reversibility,

$$\tilde{v}_3 = R(\tilde{v}_1^{p_h}), \quad \tilde{v}_4 = R(\tilde{v}_2^{p_h}) \quad \leftarrow \text{tangent to } W^s(\mathbf{0})$$

3. Finally, normalize them $v_j = \tilde{v}_j^{p_h}/\|\tilde{v}_j^{p_h}\|$, $j = 1, \ldots, 4$ and define

$$\boxed{V = \text{det}(v_1, v_2, v_3, v_4)}$$

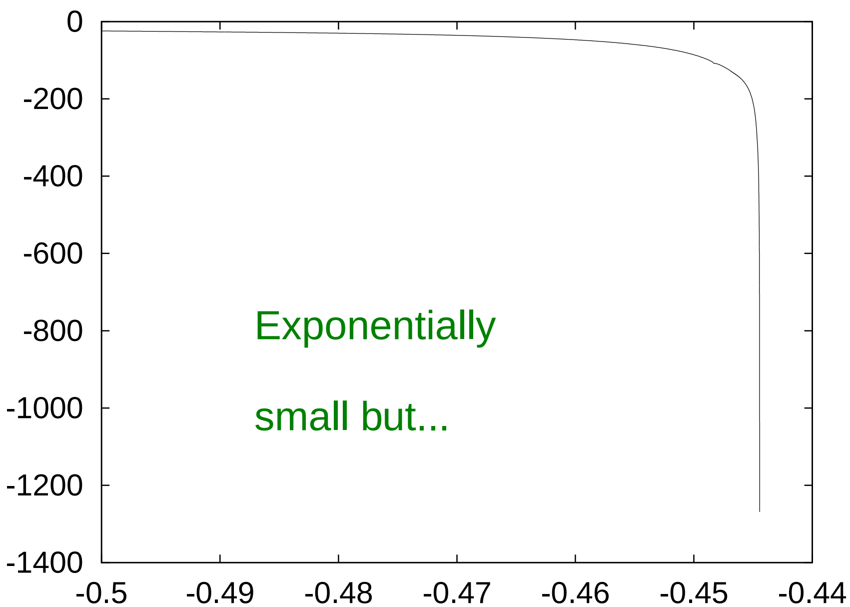
Question: How does V behave as $\epsilon \to \epsilon^c$?

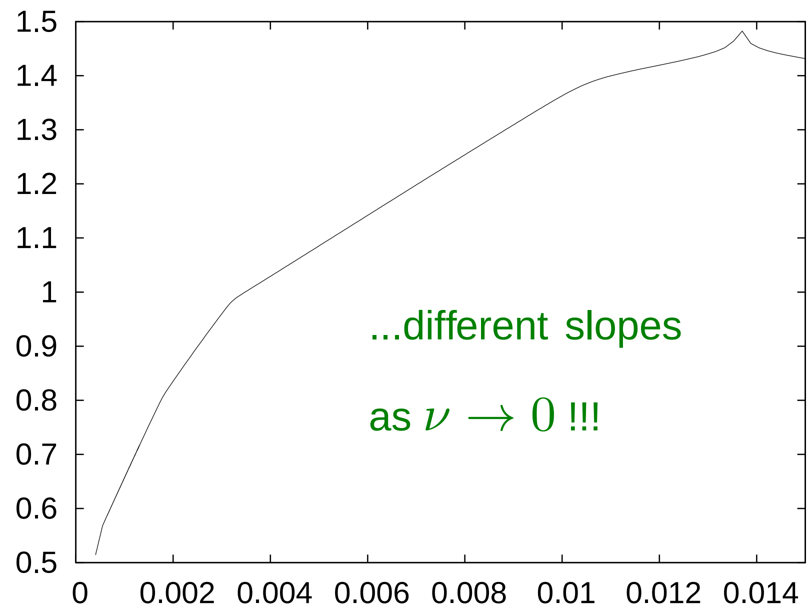
T: Behaviour of the splitting volume V

$$T: \quad \bar{\psi}_1 = \psi_1 + \delta(\bar{J}_1 + a_2\bar{J}_2), \quad \bar{\psi}_2 = \psi_2 + \delta(a_2\bar{J}_1 + a_3\bar{J}_2),$$
$$\bar{J}_1 = J_1 - \delta\sin(\psi_1), \quad \bar{J}_2 = J_2 - \delta\epsilon\sin(\psi_2).$$

We compute the volume of a $4D$ parallelotope defined by two pairs of vectors tangent to W^u and W^s at $p_h \in \Sigma$.

$a_2 = 0.5$, $a_3 = -0.75 \rightsquigarrow \epsilon^c = -4/9$. $\quad \delta = 0.5 \rightsquigarrow \theta_K = \arctan(\sqrt{23}/11)/2\pi$ ("$\in \mathbb{R}\backslash\mathbb{Q}$").





Left: $\log V$ vs. ϵ. Right: $h|\log(V)|$ vs. h $\quad (h = \log(\lambda))$.

Explanation of the behaviour of V

Consider (generic) symplectic map F_{δ_t,ϵ_t} in $\mathbb{R}^4$ that undergoes a HH bif.

Rec: δ_t: Collision angle $\hat{\theta}_{\mathrm{K}} = 2\pi(q/m + \delta_t)$. ϵ_t: Relative distance to the bifurcation.

Different (naive) important aspects:

1. "Two" exp. small effects: one within the Hamiltonian itself (already studied!), the other measures the "Hamiltonian-map distance".
2. "Two" frequencies: "Duffing" and its $2\pi\theta_{\mathrm{K}}$-perturb. + **"time" frequency**.
3. The Hamiltonian part is known $\Rightarrow$ only necessary to measure the second effect. **But:** We have a "privilegiated direction" (the time!) $\Rightarrow$ we will use an **energy function** ψ to measure the splitting in that direction (instead of using the splitting potential or the Melnikov vector which measures both effects together).

Towards a sharp upper bound of the splitting

Let F_{ϵ_t} be a family of symplectic maps s.t. at $\epsilon_t = 0$ undergoes a HH bifurcation. The inv. manifolds $W^{u/s}(\mathbf{0})$ are given by $u(\alpha, t)$ and $v(\alpha, t)$ resp., where $(\alpha, t) \in [t_0, t_0 + h) \times S^1$.

Under *reasonable* conditions: Define $\psi(\alpha, t) = E(u(\alpha, t))$, E analytic energy function. Then:

(i) **Rational Krein collision.** Let $\theta_{\mathrm{K}} = p/q$, with $(p, q) = 1$. Then, there exists $\epsilon_t^0 > 0$ s.t. for $\epsilon_t < \epsilon_t^0$

$$|\psi(\alpha, t)| \leq K \exp(-C/h), \qquad C, K > 0.$$

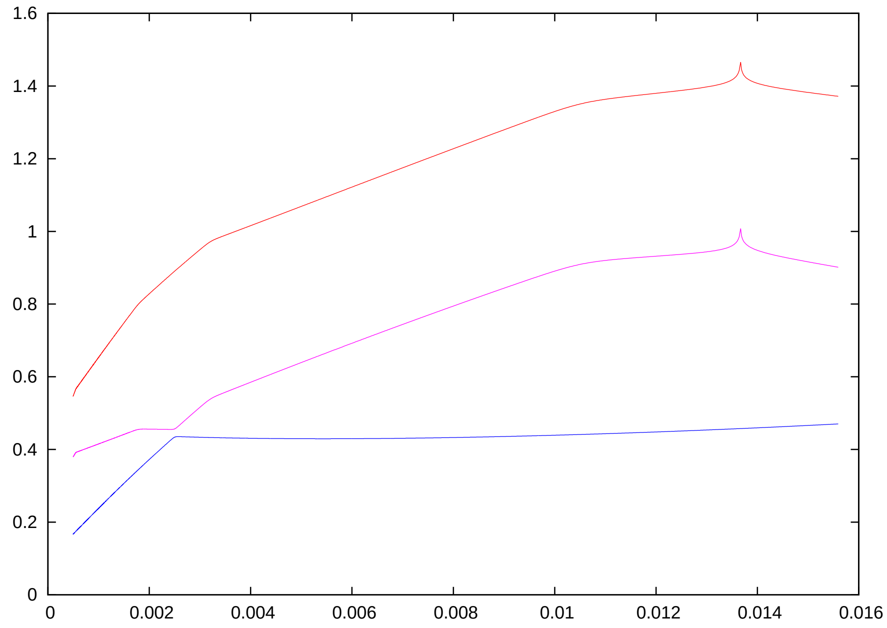
(ii) **Irrational Krein collision.** Let $\theta_{\mathrm{K}} \in \mathbb{R} \setminus \mathbb{Q}$. Then, ψ is bounded by a function that is exponentially small in a parameter γ, s.t. $\gamma \searrow 0$ when $h \searrow 0$. Moreover, the dominant harmonic $k(h)$ of ψ changes infinitely many times as $h \to 0$.

Intrinsic geometry plays a role

The theory is not fully satisfactory because (at the moment!) we can't explain:

1. **all** the different changes in slope observed.
2. **when** the changes take place.

Maximum α_M and minimum α_m angle of splitting.



$h \log(|Q|)$ vs h, being $Q = V$ (red), α_M (blue), α_m (magenta).

C. Simó, C. Valls, *A formal approximation of the splitting of separatrices in the classical Arnold's example of diffusion with two equal parameters*, Nonlinearity 14, 2001.

Part 3

Let me consider a “similar” but somehow “easier” setting:

3. Periodically perturbed Hamiltonian-Hopf

The system

We consider

$$H(\mathbf{x}, \mathbf{y}, t) = H_0(\mathbf{x}, \mathbf{y}) + \epsilon H_1(\mathbf{x}, \mathbf{y}, t), \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2,$$

where

1. H_0 is the truncated 2-dof Sokolskii NF up to order 4

$$H_0(\mathbf{x}, \mathbf{y}) = \Gamma_1 + \nu(\Gamma_2 - \Gamma_3 + \Gamma_3^2),$$

   where $\Gamma_1 = x_1 y_2 - x_2 y_1$, $\Gamma_2 = (x_1^2 + x_2^2)/2$ and $\Gamma_3 = (y_1^2 + y_2^2)/2$.

2. H_1 is periodic in t. We choose $H_1 = y_1^5 \frac{1}{5 - \sin(\gamma t)}$.

Then: $F_1 = \Gamma_1$ and $F_2 = \Gamma_2 - \Gamma_3 + \Gamma_3^2$ are first integrals of the unperturbed system H_0. We can use them to measure the splitting when perturbing.

The unperturbed system

The 2-dimensional $W^{u/s}(\mathbf{0})$ of the **unperturbed system** ($\epsilon = 0$):

1. Are contained in the level $F_1^{-1}(0)$ and $F_2^{-1}(0)$.
2. Intersect the Poincaré section $\Sigma = \max\{y_1^2 + y_2^2\}$ in the curve $x_1 = 0$, $y_1^2 + y_2^2 = 2$.
3. Are foliated by the 1-parameter family of homoclinic orbits given by $(R_1(t)\cos(\psi), R_1(t)\sin(\psi), -R_2(t)\cos(\psi), -R_2(t)\sin(\psi))$, where

   $\psi = t + \psi_0$, $\psi_0 \in [0, 2\pi)$ initial phase

   $$R_1(t) = \sqrt{2}\,\mathrm{sech}(\nu t)\tanh(\nu t), \qquad R_2(t) = \sqrt{2}\,\mathrm{sech}(\nu t).$$

   $\Longrightarrow$ Singularities at $t = (2m+1)i\pi/2\nu$, $m \in \mathbb{Z}$.

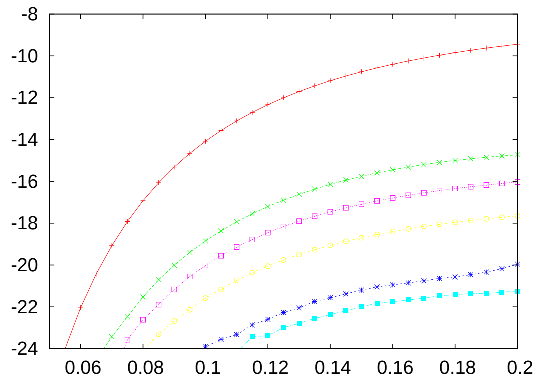
Adding an autonomous perturbation

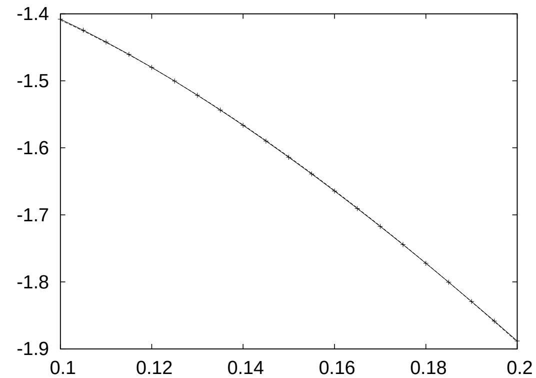
Consider $H_1(\mathbf{x}, \mathbf{y}) = y_1^5$. **Question:** Splitting behaviour w.r.t ν?

Taking suitable i.c. on W^u and on W^s we propagate them until Σ.

Let $\theta = \arctan(y_2/y_1)$. We fit numerically $F_1^{W^{u/s}}(\theta) = \sum_{k=1}^{6} a_k^{u/s}(\nu) e^{ik\theta}$.

Then we compute the difference $\Delta F_1 = F_1^{W^u} - F_1^{W^s}$.





Left: Log of the amplitudes A_i of the 6 main harmonics of ΔF_1 (vs. ν). Right: Fit of $\nu \log(A_1)$ by $f(\nu) = a\nu + b\nu\log(\nu) + c$, gives $c \approx \pi/2$ and $b \approx -5$.

Melnikov prediction: $\Delta F_1 = \mathcal{O}(\nu^{-5}\exp(-\pi/2\nu))$.

Adding a **non-***autonomous perturbation*

Consider $H_1(\mathbf{x},\mathbf{y},t) = y_1^5/(5-\cos(\gamma t))$, $\gamma = (\sqrt{5}-1)/2$, $\epsilon = 10^{-3}$.

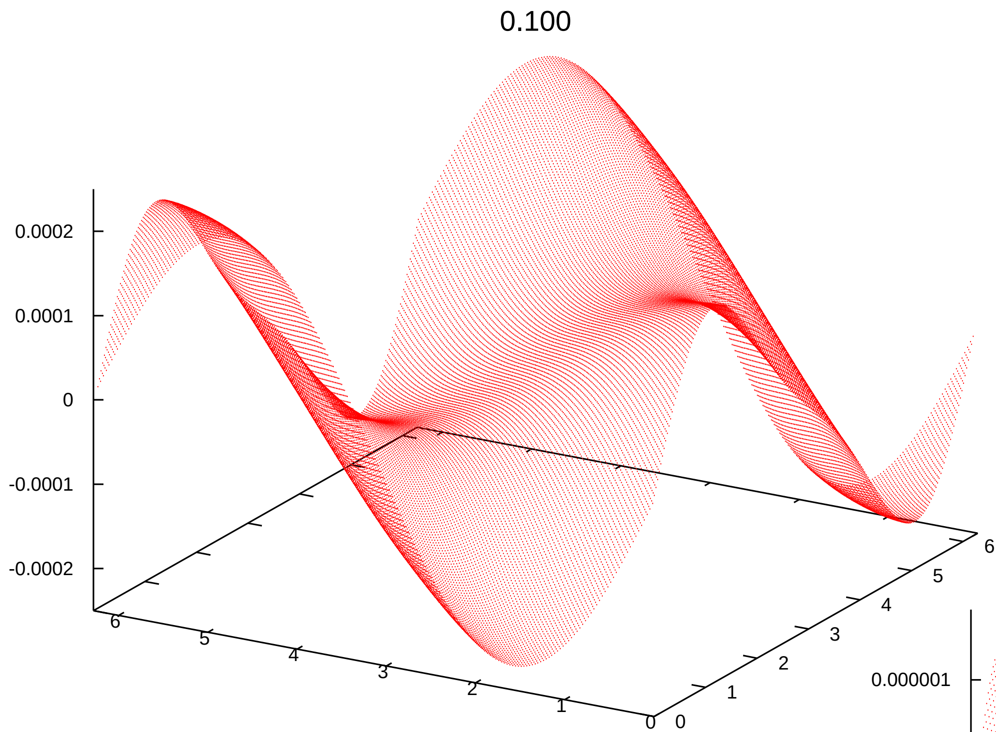
Same question: Splitting behaviour w.r.t ν?

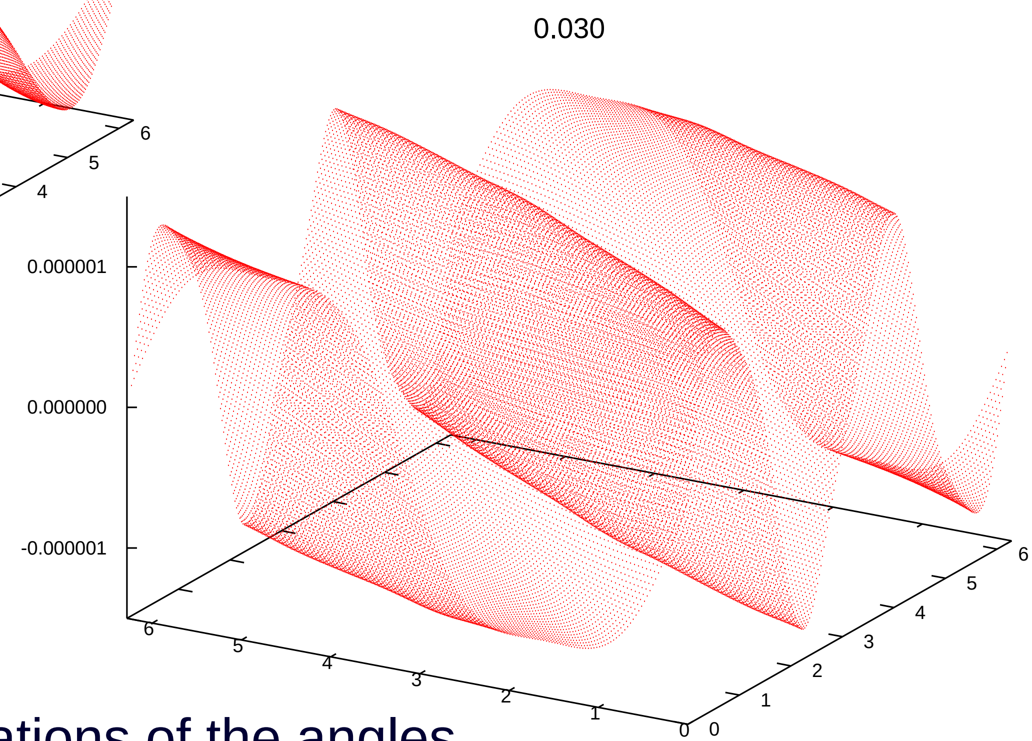
Taking suitable i.c. on W^u and on W^s, depending on the initial values of ψ_1 and the phase of γt, we propagate them up to Σ.

Similar to what was done before we compute $\Delta F_i, i=1,2$, (i.e., the splitting function) which depend on two angles, and compute the nodal curves (i.e. the zero level curves) of $\Delta F_i, i=1,2$.

Remark: Intersections of the two nodal curves $\leftrightarrow$ homoclinic trajectories.

The F_1-difference of the invariant manifolds





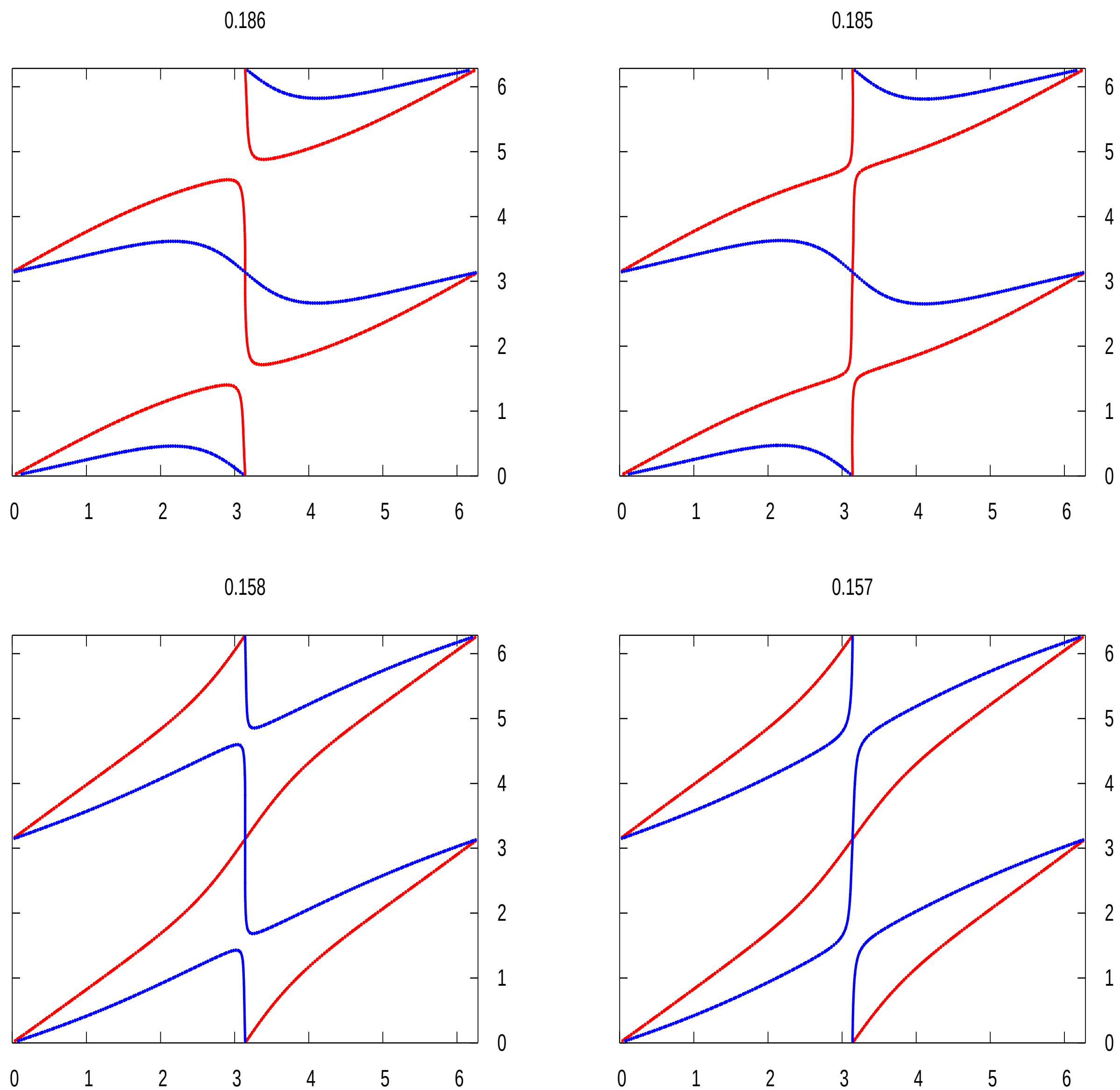
Behaviour:

$\Delta F_i \approx \sum A_j(\nu) \exp(-\beta_j \pi / 2\nu)$

where β_j contain suitable combinations of the angles.

Remark: For the displayed values of ν ΔF_2 is "almost" equal. The nodal lines "almost coincide".

Nodal curves I



resonance (k, l)

dominant term

$k\theta - l\hat{t} \approx 0$

ΔF_1 ΔF_2

left: (0,1), (0,1)

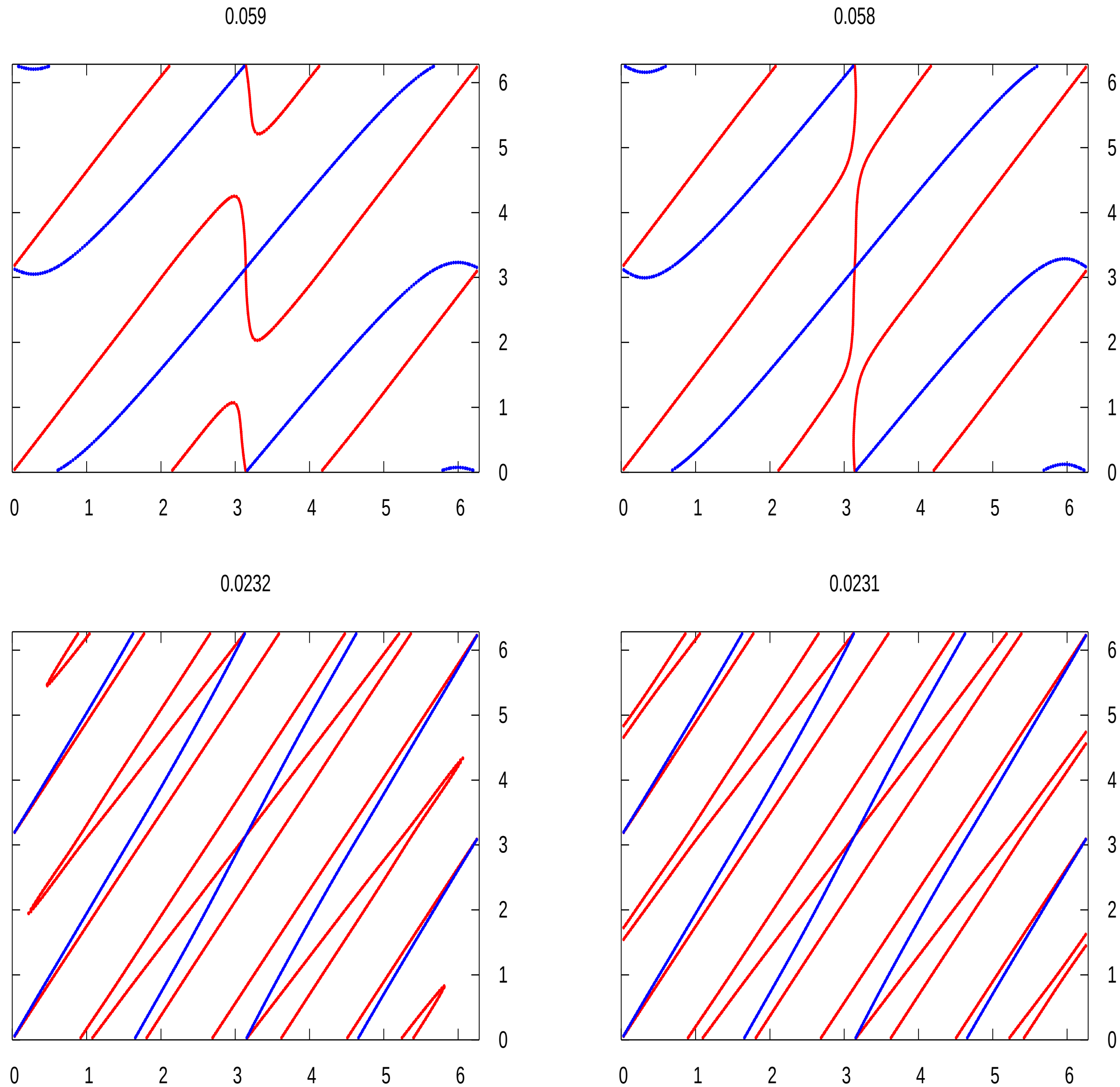
right: (1,1), (0,1)

ΔF_1 ΔF_2

left: (1,1), (0,1)

right: (1,1), (1,1)

Nodal curves II



resonance

dominant term

$k\theta - l\hat{t} \approx 0$

$\color{red}{\Delta F_1}$ $\color{blue}{\Delta F_2}$

left: (1,1), (1,1)

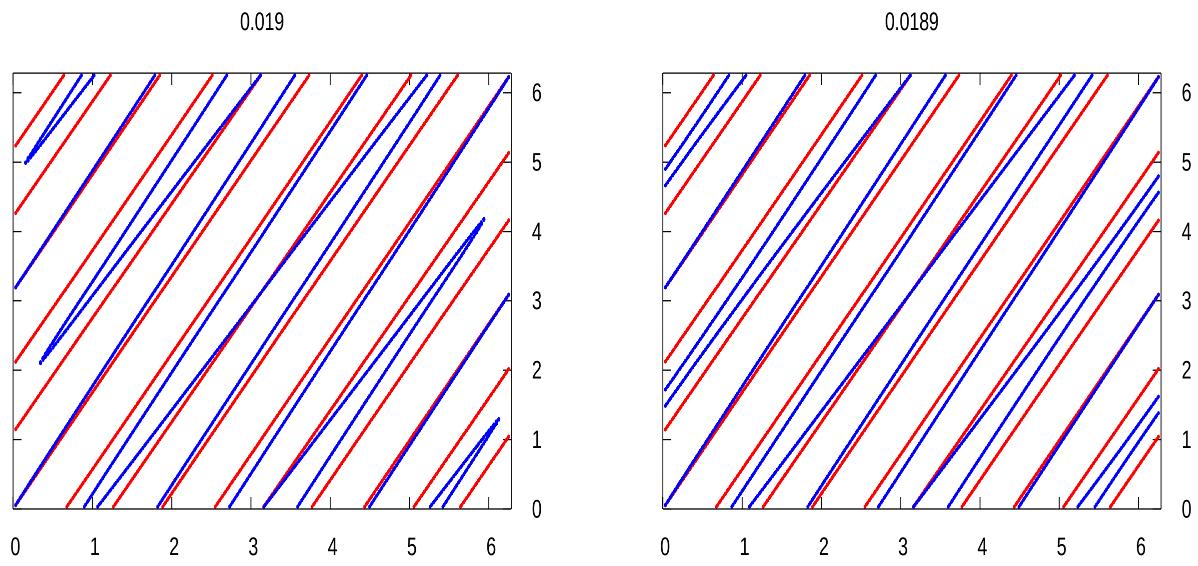
right: (2,1), (1,1)

$\color{red}{\Delta F_1}$ $\color{blue}{\Delta F_2}$

left: (2,1), (2,1)

right: (5,3), (2,1)

Nodal curves III



resonance

dominant term

$k\theta - l\hat{t} \approx 0$

ΔF_1 ΔF_2

left: (5,3), (2,1)

right: (5,3), (5,3)

Conclusions:

- Several bifurcations are observed.
- The changes in the nodal lines of ΔF_1 and ΔF_2 occur for different ν.
- The dominant terms of ΔF_1 and ΔF_2 coincide for values in between the changes.

Future work

1. Dynamics close to separatrices? Derive a return **separatrix map**.

   Ingredients:

   (a) The splitting function.

   (b) Flight times (in $\hat{t}$ and θ) from Σ to Σ.

2. Use the separatrix map to obtain **quantitative** information about distance to maximal tori, to secondary resonances,... from the separatrices.

3. Analyse the diffusion properties by performing accurate computations in different regimes.

Thanks for your attention!!