Periodically perturbed Hamiltonian-Hopf.

Dynamics, Bifurcations, and Strange Attractors

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The system

In this presentation we consider the system

$$H(x_1, x_2, y_1, y_2, t) = H_0(x_1, x_2, y_1, y_2) + \epsilon H_1(x_1, x_2, y_1, y_2, t),$$

where

$$H_0 = x_1 y_2 - x_2 y_1 + \nu \left(\frac{x_1^2 + x_2^2}{2} + \frac{y_1^2 + y_2^2}{2} \left(-1 + \frac{y_1^2 + y_2^2}{2} \right) \right),$$

and

$$H_1 = \frac{y_1^5}{(d - y_1)(c - \cos(\theta))}, \quad \theta = \gamma t + \beta.$$

1. We fix concrete values of c, d, γ and ϵ.
2. $\nu > 0$ is a perturbative parameter.
3. The parameter $\beta \in [0, 2\pi)$ is the initial time phase.

Contents

Why this (2+1/2)-dof Hamiltonian system?

2-dof Hamiltonian Hopf (HH): Sokolskii NF

2-dof HH codim 1: Consider a 1-param. family of 2-dof Hamiltonians H_δ undergoing a HH bifurcation (at the origin).

Concretely: for $\delta > 0$ elliptic-elliptic, $\delta < 0$ complex-saddle.

Analysis of the HH bifurcation $\rightarrow$ Reduction to **Sokolskii NF**:

1. Taylor expansion at $\mathbf{0}$: $H_\delta = \sum_{k\geq 2} \sum_{j\geq 0} \delta^j H_{k,j}$, where $H_{k,j} \in \mathbb{P}_k$ homogeneous polynomial of order k.
2. Williamson NF (double purely imaginary eigenvalues $\pm i\omega$): $H_{2,0} = -\omega(x_2 y_1 - x_1 y_2) + \frac{1}{2}(x_1^2 + x_2^2)$.
3. Use Lie series to order-by-order simplify $H_{2,j}, j > 1$ and $H_{k,j}, k > 2, j > 0$.

   But: **non-semisimple** linear part!

   Then, at each order (k, j), one looks for $G \in \mathbb{P}_k$ s.t.

$$H_{k,j} + \mathrm{ad}_{H_2}(G) \in \mathrm{Ker}\,\mathrm{ad}_{H_2}^{\top}.$$

2-dof HH: Sokolskii NF

4. Introducing the Sokolskii coordinates $(dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = dR \wedge dr + d\Theta \wedge d\theta)$

$$y_1 = r\cos(\theta),\ y_2 = r\sin(\theta),\ R = (x_1 y_1 + x_2 y_2)/r,\ \Theta = x_2 y_1 - x_1 y_2,$$

one has $H_2^{\top} = -\omega\Theta + \frac{1}{2}r^2$ and

$$\mathsf{NF}(H_\delta) = \omega\Gamma_1 + \Gamma_2 + \sum_{\substack{k,l,j\geq 0\\ k+l\geq 2}} a_{k,l,j}\,\Gamma_1^k\,\Gamma_3^l\,\delta^j, \qquad \leftarrow \text{formal}$$

where

$$\Gamma_1 = x_1 y_2 - x_2 y_1,\ \Gamma_2 = (x_1^2 + x_2^2)/2 \text{ and } \Gamma_3 = (y_1^2 + y_2^2)/2.$$

Γ_1 is a (formal) first integral, hence $W^{u/s}(\mathbf{0}) = \{\Gamma_1 = 0\} \cap \{\mathsf{NF}(H_\delta) = 0\} = \{\Gamma_2 + \delta\, a_{0,1,1}\Gamma_3 + a_{0,2,0}\Gamma_3^2 + \mathcal{O}(\delta^2\Gamma_3, \delta\Gamma_3^2, \Gamma_3^3) = 0\}$.
$W^{u/s}(\mathbf{0})$ real $\Leftrightarrow \delta a_{0,1,1} < 0$. Moreover,

- If $a_{0,2,0} > 0$ they bound a finite domain of size $\Gamma_2 = \mathcal{O}(\delta^2), \Gamma_3 = \mathcal{O}(\delta)$.
- If $a_{0,2,0} < 0$ they are unbounded.

The unperturbed model

We consider the bounded case.

Introducing $\delta = -\nu^2$, and rescaling $x_i = \nu^2 \tilde{x}_i, \omega y_i = \nu\, \tilde{y}_i, i = 1, 2,$ $\omega t = \tilde{t}$, one has (skipping ˜ from the new variables)

$$\boxed{\text{NF}(H_\delta) = \Gamma_1 + \nu \left(\Gamma_2 + a\Gamma_3 + \eta\Gamma_3^2\right) + \mathcal{O}(\nu^2)}$$

where $a = -a_{0,1,1}/\omega^2$ and $\eta = a_{0,2,0}/\omega^4$.

Taking $a = -1, \eta = 1$, and truncating $\rightsquigarrow$ the unperturbed system.

Geometry of $W^{u/s}(\mathbf{0})$: In polar coord $x_1 + i x_2 = R_1 e^{i\psi_1}$, $y_1 + i y_2 = R_2 e^{i\psi_2}$ the restriction to (R_1, R_2)-components is a Duffing Hamiltonian system. On $W^{u/s}(\mathbf{0})$ one has $\psi_1 = \psi_2 - \pi$, $\psi_2 = t + \psi_0$, and they are are foliated by homoclinic orbits

$$x_1(t) + i x_2(t) = -R_1(t) e^{i\psi}, \; y_1(t) + i y_2(t) = R_2(t) e^{i\psi},$$

being $\psi = t + \psi_0$, $R_1(t) = \sqrt{2}\,\text{sech}(\nu t)\tanh(\nu t)$, and $R_2(t) = \sqrt{2}\,\text{sech}(\nu t)$.

The effect of a periodic forcing on H_0

Periodic forcing

We add to H_0 the periodic perturbation $\epsilon H_1 = \epsilon g(y_1) f(\theta)$ where

$$g(y_1) = y_1^5 (d - y_1)^{-1}, \qquad f(\theta) = (c - \cos(\gamma t + \beta))^{-1}.$$

Remarks:

1. Restricted to the unperturbed $W^{u/s}(\mathbf{0})$, y_1 becomes 1-periodic in t.
2. $f(\theta)$ periodic in t with frequency γ.

$\Rightarrow$ If $\gamma \in \mathbb{R}$ then quasi-periodic!

For simulations we choose $c = 5$, $d = 7$, and $\gamma = (\sqrt{5} - 1)/2$.

Recall that:

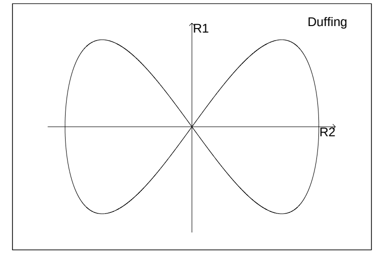
(β, ψ_0) are initial conditions on a fundamental domain (torus $\mathcal{T}$) of $W^{u/s}(\mathbf{0})$.

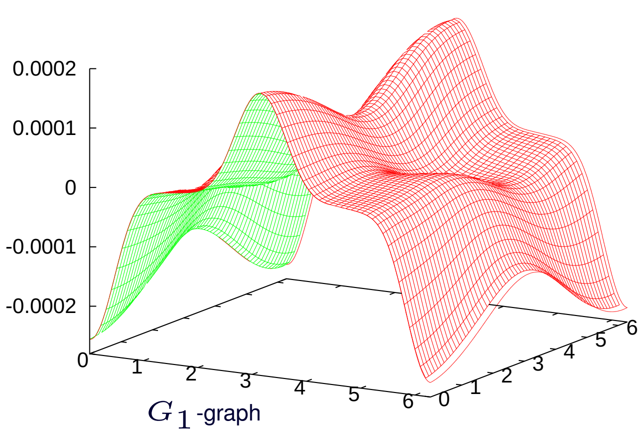
ν is a small parameter (in H_0).

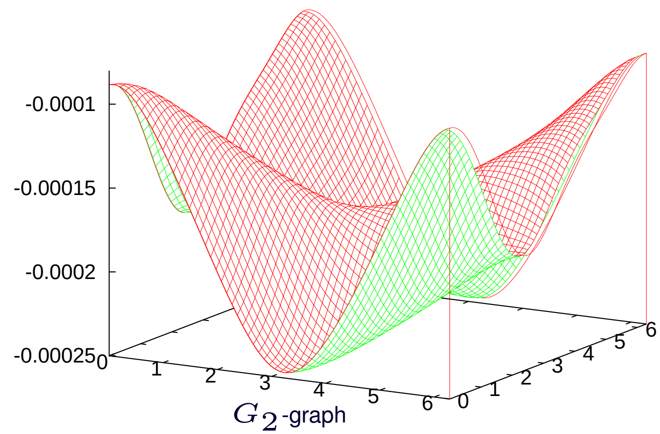
The invariant manifolds

We express $H = G_1 + \nu G_2,\ G_1 = \Gamma_1,\ G_2 = \Gamma_2 - \Gamma_3 + \Gamma_3^2$,
and we consider the Poincaré section

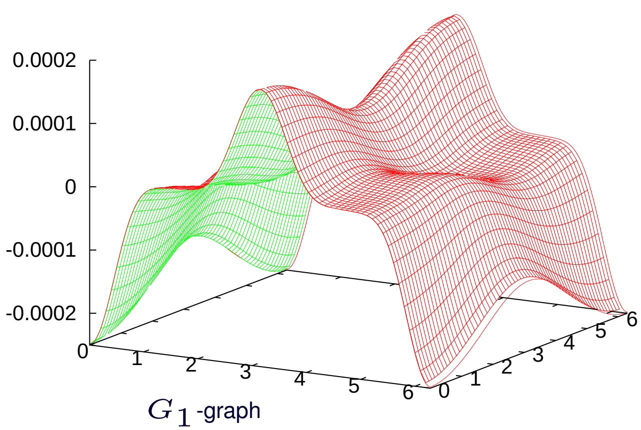
$$\Sigma = \max(R_2)$$

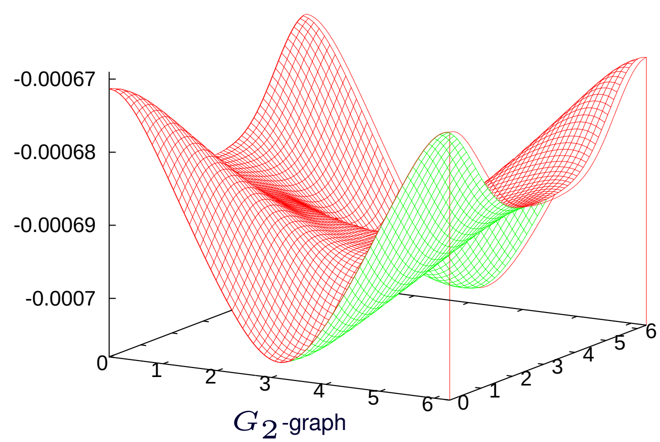






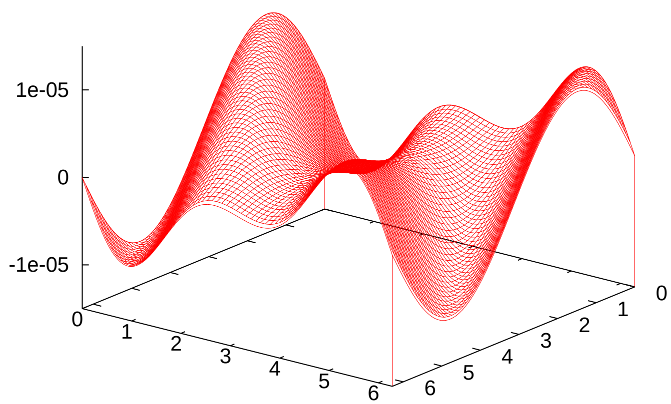
$\nu = 2^{-4}$



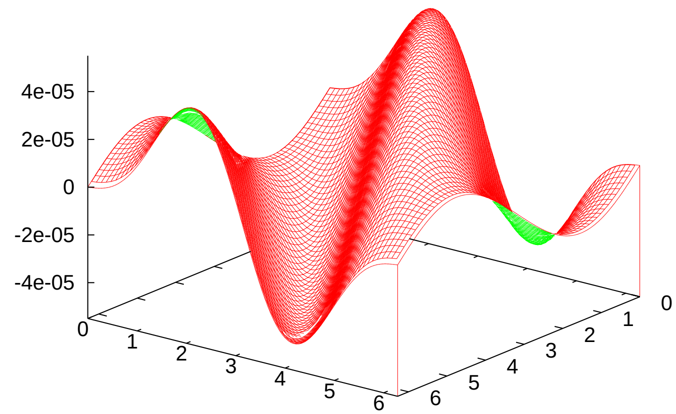


$\nu = 2^{-6}$

The splitting

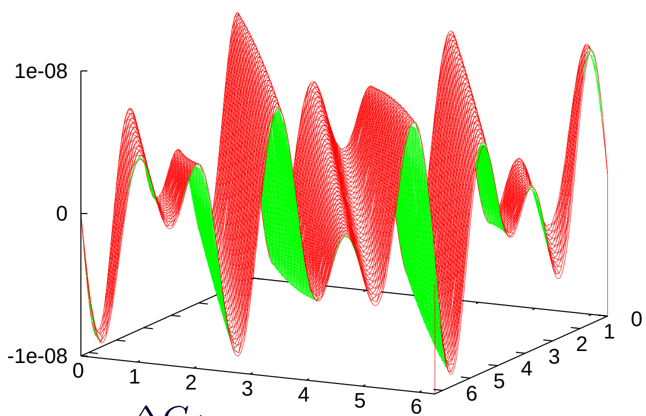


ΔG_1

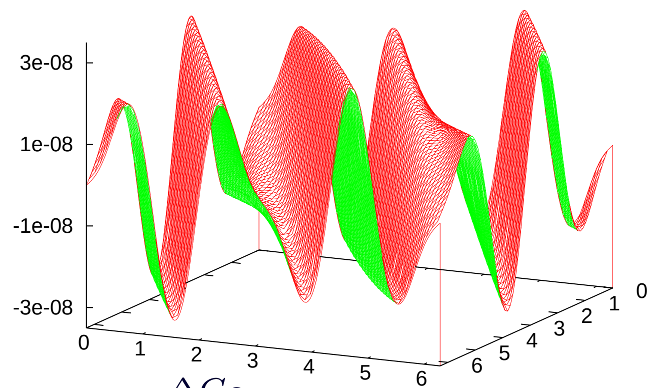


ΔG_2

$\nu = 2^{-4}$



ΔG_1



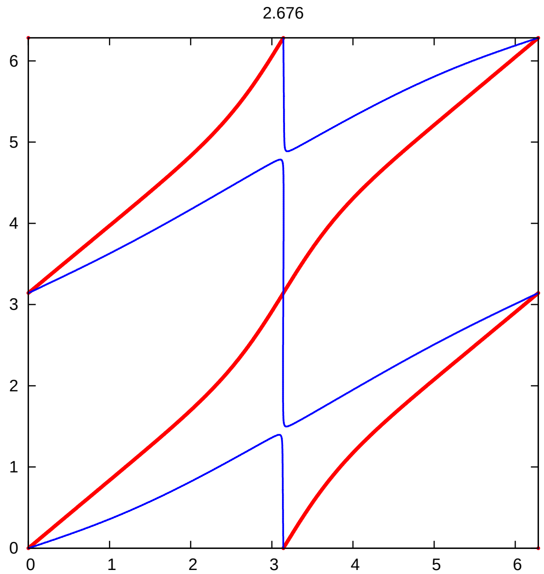
ΔG_2

$\nu = 2^{-6}$

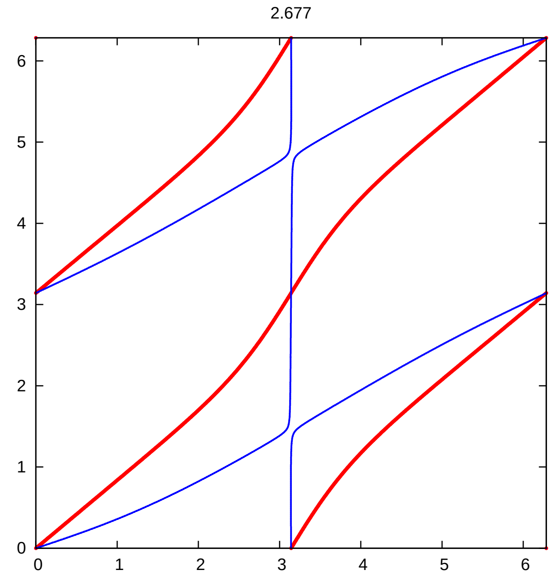
Remarks on the previous computations

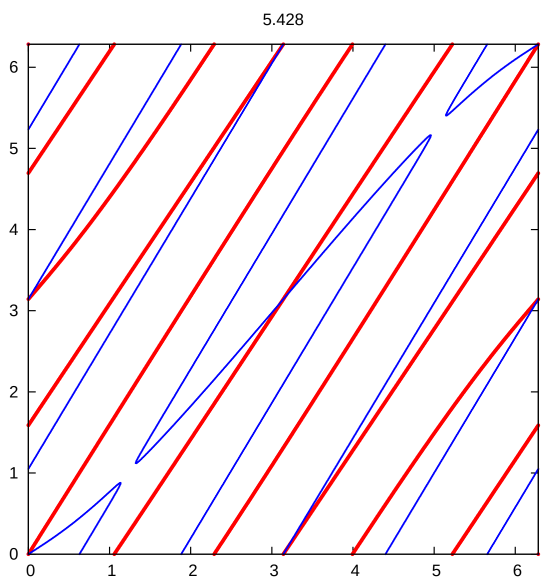
1. We propagate a set $\{\psi_{0,k}, \beta_{0,j}\}$, $0 \leq k, j \leq 512$, of initial points in $\mathcal{T}$ (i.e. a total number of 2^{18} initial conditions) up to reach the Poincaré section Σ.
2. The numerical integration is performed using an ad-hoc implemented Taylor time-stepper scheme with quadruple precision.
3. The propagation of $\mathcal{T}$ up to Σ gives a 2D torus $\mathcal{T}_\Sigma$. The invariant manifolds $W^{u/s}(\mathbf{0})$ in $\mathbb{R}^4$ are defined by the G_1 and the G_2-graphs over $\mathcal{T}_\Sigma$.
4. To compute the difference (i.e. the splitting) between $W^u(\mathbf{0})$ and $W^s(\mathbf{0})$ we need to compare them at the same points of $\mathcal{T}_\Sigma$. Hence, we select a mesh of angles ψ and β within $\mathcal{T}_\Sigma$, and refine the initial conditions in $\mathcal{T}$ using a Newton method.

Nodal lines: bifurcations

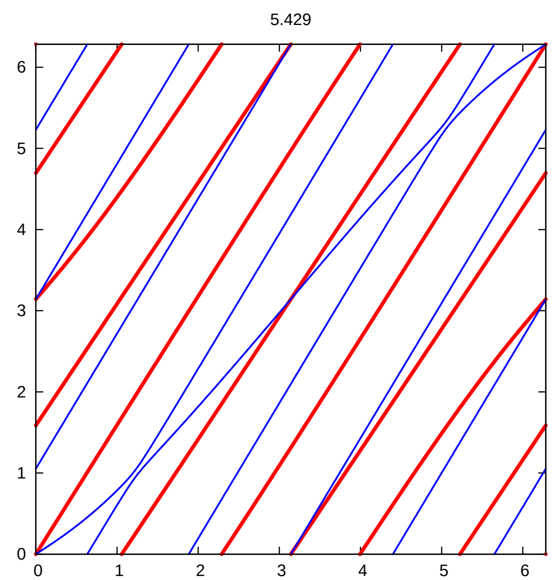


(1,1), (1,0) ⟶ (1,1), (1,1)





(2,3), (1,2) ⟶ (2,3), (2,3)



Change of dominant harmonics

$-\log_2 \nu_2$	$-\log_2 \nu_1$	Change of the dom harm of the G_1, G_2-splittings
2.443	2.444	(1,0), (1,0) $\longrightarrow$ (1,1), (1,0)
2.676	2.677	(1,1), (1,0) $\longrightarrow$ (1,1), (1,1)
4.112	4.113	(1,1), (1,1) $\longrightarrow$ (1,2), (1,1)
4.300	4.301	(1,2), (1,1) $\longrightarrow$ (1,2), (1,2)
5.133	5.134	(1,2), (1,2) $\longrightarrow$ (2,3),(1,2)
5.428	5.429	(2,3), (1,2) $\longrightarrow$ (2,3), (2,3)
6.234	6.235	(2,3), (2,3) $\longrightarrow$ (3,5), (2,3)

Table 1: Changes in the dominant harmonic of the G_1 splitting function and the G_2 splitting function. The bifurcation takes place for $\nu \in (\nu_1, \nu_2)$.

Theoretical/symbolical results

The Melnikov integral

For simplicity, we discuss on the G_1-splitting (similar for the G_2-splitting).

Recall that $H_1 = g(y_1)f(\theta)$ where

$g(y_1) = y_1^5(d - y_1)^{-1} \rightsquigarrow g'(y_1) = \sum_{k \geq 0} d_k y_1^{4+k}$,

$f(\theta) = (c - \cos(\theta))^{-1} = \sum_{j \geq 0} c_j \cos(j\theta)$.

Then, at first order in ϵ, the variational equation is given by

$$\frac{dG_1}{dt} = \epsilon\{G_1, H_1\} = \epsilon y_2 \sum_{k \geq 0} d_k y_1^{4+k} f(\theta)$$

Melnikov: The distance $G_1^u(\psi_0, \beta) - G_1^s(\psi_0, \beta) = \epsilon \Delta G_1 + \mathcal{O}(\epsilon^2)$, is given by

$$\Delta G_1 = 4\epsilon \int_{-\infty}^{\infty} \sin(t + \psi_0)\, f(\gamma t + \beta) \sum_{k \geq 0} \frac{\sqrt{2^{k+1}}\, d_k\, (\cos(t + \psi_0))^{4+k}}{(\cosh(\nu t))^{5+k}} dt,$$

Recall that on the unperturbed separatrices $\psi = t + \psi_0$, $\theta = \gamma t + \beta$, $(\psi_0, \beta) \in \mathcal{T}$.

Comparison numerics/symbolic evaluation

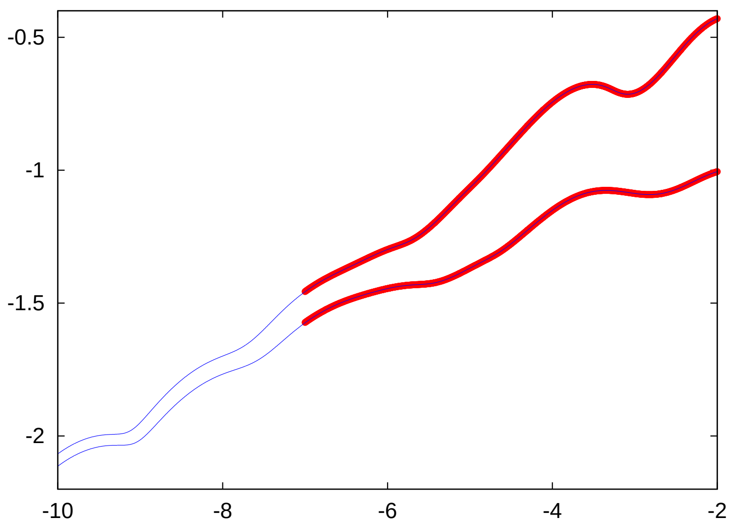
After some algebra one obtains

$$\Delta G_1 = \epsilon \sum_{j\geq 0} c_j \sum_{k\geq 0} 2^{\frac{3+k}{2}} d_k \sum_{0\leq 2i\leq 4+k} b_{4+k,i} \sum_{l=\pm 1} I_1 \sin((k+5-2i)\psi_0 + lj\beta),$$

where

$$I_1 = I_1(k+5-2i+lj\gamma, \nu, k+5), \qquad I_1(s,\nu,n) = \int_{\mathbb{R}} \frac{\cos(st)}{(\cosh(\nu t))^n}\, dt$$

$$b_{m,i} = \frac{1}{2^m(m+1)} \binom{m+1}{i} (m+1-2i)$$



We represent $\log(\Delta G_i/\epsilon)\sqrt{\nu}$, for $i=1$ (bottom) and $i=2$ (top), as a function of $\log_2(\nu)$.

Red: Direct numerical computations.

Blue: Sum of the significant terms of the Melnikov series.

The main result

For the system $H = H_0 + \epsilon H_1$ under consideration, consider:

1. $\epsilon > 0$, $c > 1$, $d > \sqrt{2}$, $\nu < \nu_M << 1$ small enough,
2. $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ a quadratic number ($\exists C > 0$, $\left|\gamma - \frac{p}{q}\right| \geq \frac{C}{q^2}, \forall p/q \in \mathbb{Q}$).

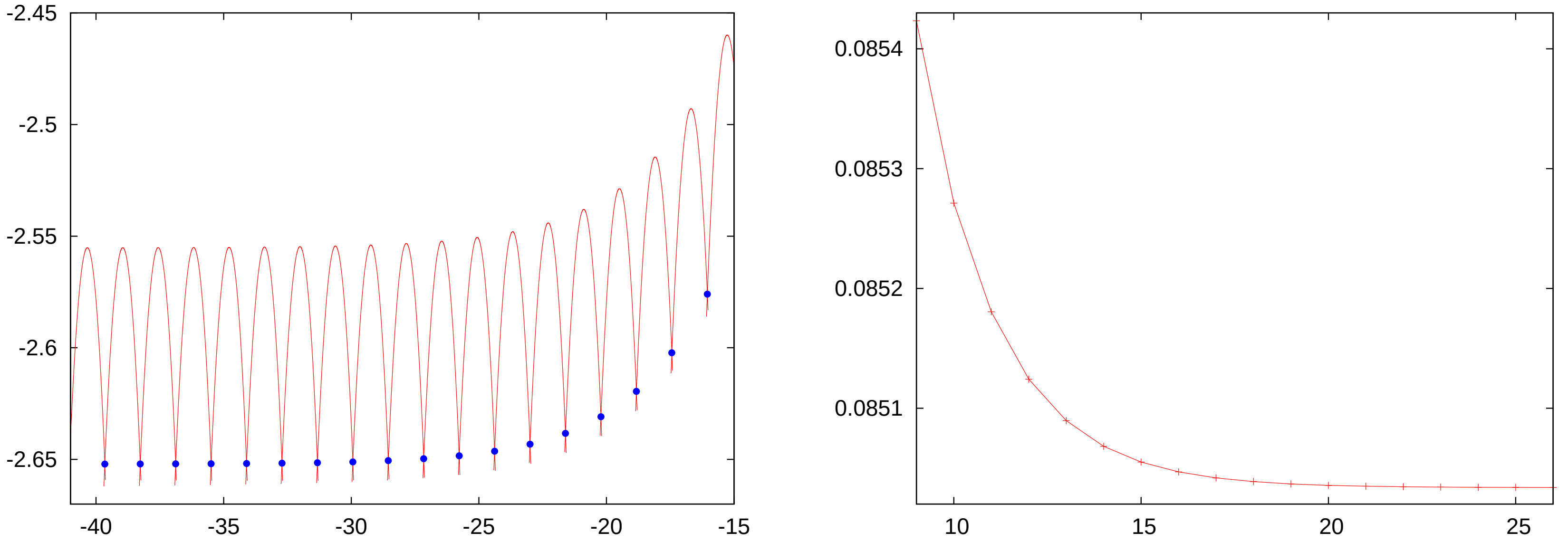
Denote by m_1/m_2 a best approximant of γ, and assume that it corresponds to the dominant harmonic in ΔG_1 (resp. ΔG_2) for $\nu \in (\nu_0, \nu_1)$, $\nu_0, \nu_1 < \nu_M$. Let $c_s \in \mathbb{R}$ be the constant such that

$$|m_1 - \gamma m_2| = \frac{1}{c_s m_1}.$$

There exists a "universal" function $\psi_1(L)$ (resp. $\psi_2(L)$) depending on $L = \nu m_1^2 c_s$ (but not depending on c_s and ν explicitly!) such that, for $\nu \in (\nu_0, \nu_1)$,

$$\Delta G_i \approx e^{\frac{-\psi_i(L)}{\sqrt{\nu}}}, \quad i = 1, 2.$$

Changes of the dominant harmonic

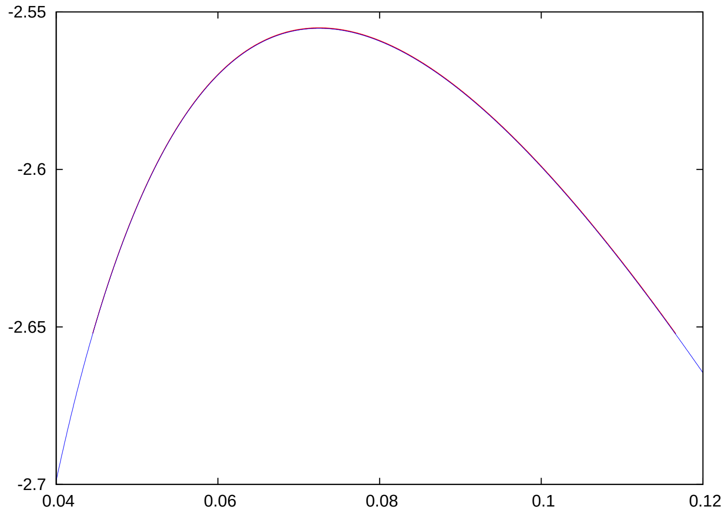


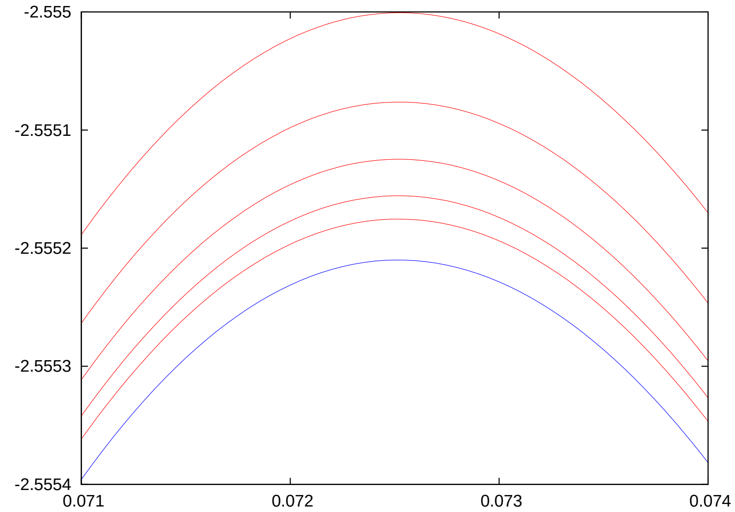
Left: $\gamma = (\sqrt{5}-1)/2$, $\epsilon = 10^{-4}$. We represent $\log(\Delta G_1/\epsilon)\sqrt{\nu}$ as a function of $\log_2(\nu)$. The dots correspond to the values ν_j where changes the dominant harmonic (from $m_1 = F_j \to F_{j+1}$, where $\{F_j\}_j$ denotes the Fibonacci sequence). The rightmost change corresponds to $m_1 = 55 \to m_1 = 89$, while the leftmost to $m_1 = 196418 \to m_1 = 317811$.

Right: One has $\nu_{j+1} \sim \gamma^2 \nu_j$, then $\nu_j \sim \gamma^{2j} K$. We represent $\nu_j \gamma^{-2j}$, we see that $K \approx 0.0850$ for j large enough.

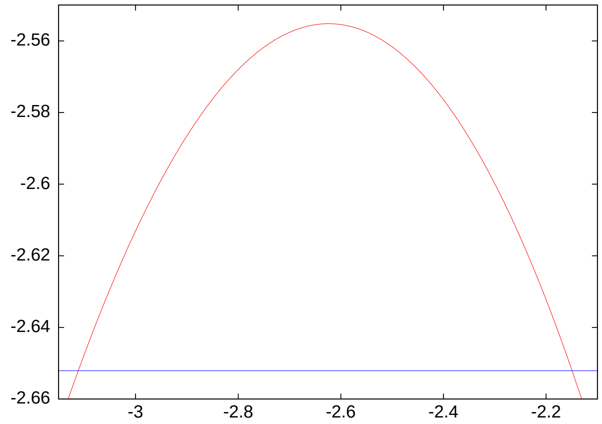
The function $\psi(L)$

We have obtained an explicit expression for the function $\psi(L)$. Denote by $\tilde{L} = L/c_s$. For $\gamma = (\sqrt{5}-1)/2$ one has $c_s \approx \sqrt{5}(1+\gamma)$.



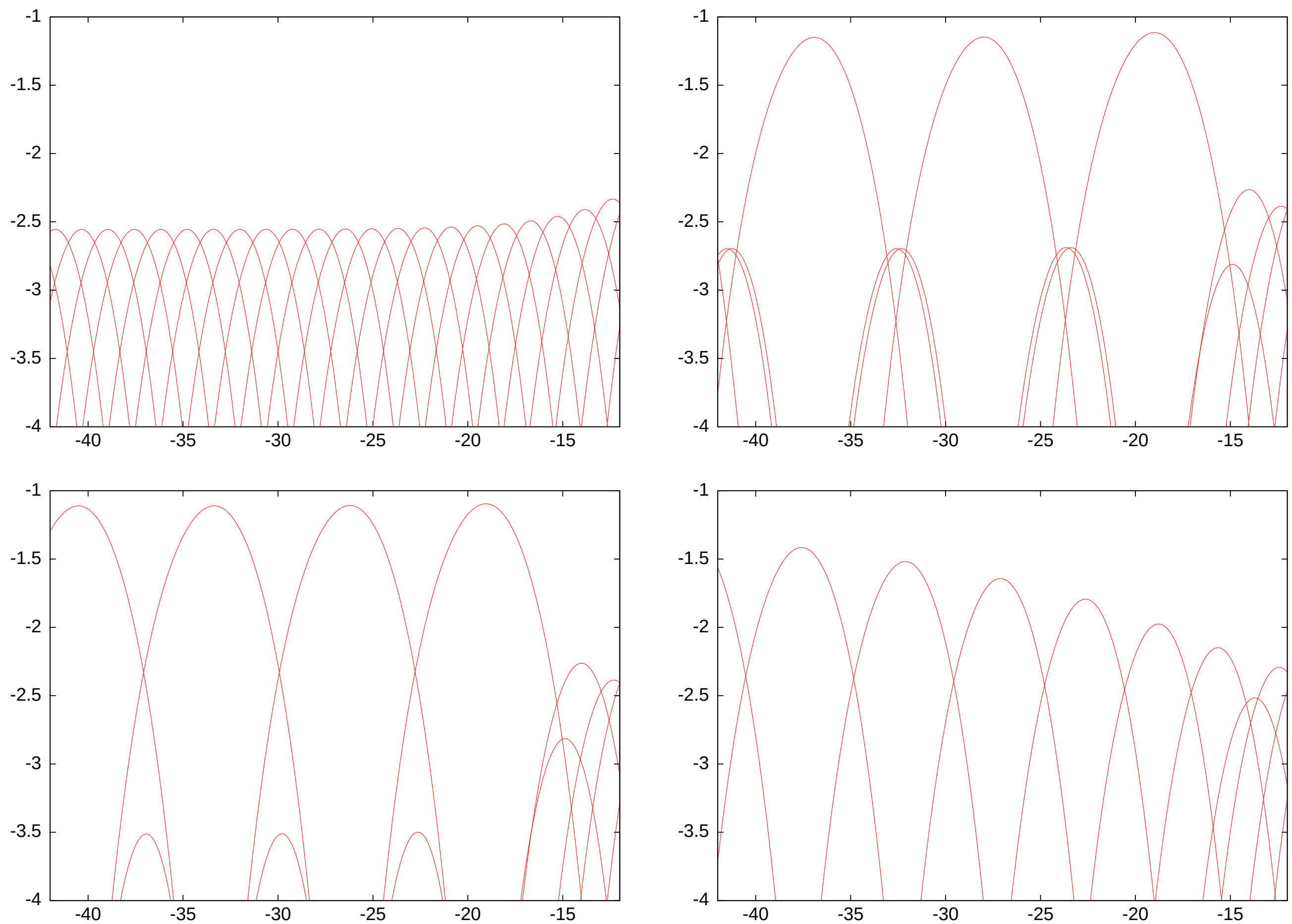


Left: Five leftmost picks of the previous fig. as a function of $\tilde{L}$ (in red). $\psi(\tilde{L})$ in blue. Right: Magnification of the central zone of the left plot. The picks tend to $\psi(\tilde{L})$ as ν decreases (and m_1 increases).



$\psi(\tilde{L})$ as a function of $\log(\tilde{L})$.

Other frequencies



$\log(\Delta G_1)/\epsilon\sqrt{\nu}$ as a function of $\log_2(\nu)$. Top left : $\gamma = (\sqrt{5}-1)/2 = [1, 1, 1, 1, 1, ...]$.

Top right: $\gamma = [10 \times 1, 1, 10, 1, 1, 10, 1, 1, 10, 1, ...] \approx 0.6180512268192526496794$.

Bottom left : $\gamma = [10 \times 1, 1, 10, 1, 10, 1, 10, 1, 10...] \approx 0.6180513744611582707944$.

Bottom right: $\gamma = [10 \times 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ...] \approx 0.6180206632934375446297$.

Conclusions and future work

We have studied...

1. the splitting of the invariant manifolds after a Hamiltonian-Hopf bifurcation when a periodic forcing is acting on the system. The role of the internal and forcing frequencies has been clarified: they lead to a quasi-periodic effect.
2. the asymptotic behavior of the splitting. In particular, we have determined the changes of dominant harmonic in the asymptotic behavior. **All the quotients** of the continuous fraction of γ play a role: they determine which frequencies are observed in the exponent of the splitting behavior.

Future work:

1. Construct a 4D (adapted) separatrix map (**passage time** close to the complex-saddle point!).
2. Geometry of the phase space (resonance web) and diffusive properties.
3. Analogous 4D symplectic map case (rational/irrational Krein collision).